Wave induced second order hydroelastic response of the vertical circular cylinder partially filled with liquid

Ouled Housseine C.(1), **Choi Y.M.** (1), **Malenica S.** (1) **& Kwon S.H.**(2)
(1)*Bureau Veritas Marine & Offshore, Paris, France*
(2)*Pusan National University, Busan, South Korea*

Introduction

The hydroelastic response of the vertical circular cylinder partially filled with water is considered. The outer problem is formulated up to 2nd order while the inner sloshing problem is considered linear. The boundary value problem is solved using an appropriate eigenfunction expansion technique. Basic configuration is shown in Figure 1.

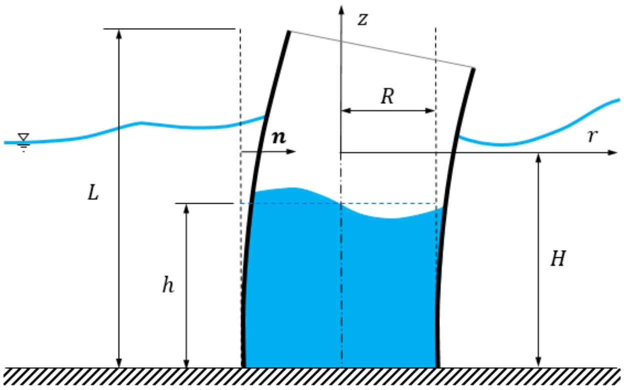

Figure 1: Basic configuration

The theoretical model for the fluid part is based on potential flow assumptions for both exterior and interior domains. On the structural side the simplified beam model is used. In principle, the beam model might not be the most appropriate for large columns with relatively thin cylindrical structure. However, since the purpose of the present work is to validate the coupling procedure and to provide reference results for the validation of the numerical codes, the structural problem has been simplified. The more advanced membrane type of structural modelling was discussed in [6] within the context of the closed flexible fish cages. Even if the membrane structural model leads to a more complex structural mode shapes, the basic principles of hydroelastic coupling, using the eigenfunction expansion methodology proposed here, remain the same.

Hydrodynamic problems

Exterior domain

Up to second order the body motion/deformation vector $\{\boldsymbol{H}\}$ is defined in the form of a series of the modal contributions:

$$\mathbf{H}(\mathbf{x},t) = \varepsilon\mathbf{H}^{(1)}(\mathbf{x},t) + \varepsilon^2\mathbf{H}^{(2)}(\mathbf{x},t) = \sum_{i=1}^{N}\left[\varepsilon X_i^{(1)}(t) + \varepsilon^2 X_i^{(2)}(t)\right]\mathbf{h}_i(\mathbf{x}) \tag{1}$$

where $\boldsymbol{h}_i(\boldsymbol{x})$ is the space dependent modal shape vector and $X_i^{(1)}(t)$ are the time dependent modal amplitudes.
At the same time, the total velocity potential $\Phi(\boldsymbol{x},t)$ is also decomposed into the first order and the second order parts:

$$\Phi(\mathbf{x},t) = \varepsilon\Phi^{(1)}(\mathbf{x},t) + \varepsilon^2\Phi^{(2)}(\mathbf{x},t) \tag{2}$$

The corresponding boundary conditions at the free surface and the body, at different orders become:

$$\frac{\partial^2\Phi^{(1)}}{\partial t^2} + g\frac{\partial\Phi^{(1)}}{\partial z} = 0 \tag{3}$$

$$\frac{\partial^2\Phi^{(2)}}{\partial t^2} + g\frac{\partial\Phi^{(2)}}{\partial z} = -2\nabla\Phi^{(1)}\nabla\frac{\partial\Phi^{(1)}}{\partial t} + \frac{1}{g}\frac{\partial\Phi^{(1)}}{\partial t}\left[\frac{\partial^3\Phi^{(1)}}{\partial t^2\partial z} + g\frac{\partial^2\Phi^{(1)}}{\partial z^2}\right] \tag{4}$$

$$\nabla\Phi^{(1)}\mathbf{n} = \dot{\mathbf{H}}^{(1)}\mathbf{n} \tag{5}$$

$$\nabla\Phi^{(2)}\mathbf{n} = \dot{\mathbf{H}}^{(2)}\mathbf{n} - \left[\left(\mathbf{H}^{(1)}\nabla\right)\nabla\Phi^{(1)}\right]\mathbf{n} + \left(\dot{\mathbf{H}}^{(1)} - \nabla\Phi^{(1)}\right)\mathbf{n}^{(1)} \tag{6}$$

where $\mathbf{n}^{(1)}$ is the first order correction of the normal vector due to body deformations [2].

In order to be able to solve for the amplitudes of the body motions, the total velocity potential is decomposed into the incident Φ_I , diffraction Φ_D and the radiation parts Φ_{Rj} :

$$\Phi^{(i)} = \Phi_I^{(i)} + \Phi_D^{(i)} + \sum_{j=1}^{N} \dot{X}_j^{(i)} \Phi_{Rj}^{(i)} , \quad i = 1, 2 \tag{7}$$

The incident velocity potential is known analytically and the diffracted and the radiated velocity potentials are subdivided by defining the following body and free surface boundary conditions for the radiation velocity potential:

$$\nabla\Phi_{Rj}^{(i)}\mathbf{n} = \mathbf{h}_j\mathbf{n} \quad , \quad \frac{\partial^2\Phi_{Rj}^{(i)}}{\partial t^2} + g\frac{\partial\Phi_{Rj}^{(i)}}{\partial z} = 0, \quad i = 1, 2 \tag{8}$$

The diffraction velocity potentials satisfies the remaining parts of the body and the free surface boundary conditions [2].

Interior domain

In the interior domain, the velocity potential, denoted by $\Psi(\mathbf{x}, t)$, is also decomposed into its first and second order parts:

$$\Psi(\mathbf{x}, t) = \sum_{j=1}^{N} \left[\varepsilon \dot{X}_j^{(1)} \Psi_{Rj}^{(1)} + \varepsilon^2 \dot{X}_j^{(2)} \Psi_{Rj}^{(2)}\right] \tag{9}$$

Only the radiation part of the velocity potential exists and it is defined by the following body and the free surface conditions:

$$\nabla\Psi_{Rj}^{(i)}\mathbf{n} = -\mathbf{h}_j\mathbf{n} \quad , \quad \frac{\partial^2\Psi_{Rj}^{(i)}}{\partial t^2} + g\frac{\partial\Psi_{Rj}^{(i)}}{\partial z} = \zeta_j^A, \quad i = 1, 2 \tag{10}$$

where ζ_j^A is the vertical motion of the waterplane area A_w for each mode, given by $\zeta_j^A = \iint_{S_T} \mathbf{h}_j\mathbf{n}dS / A_w$ (see [1] or [4]).

Frequency domain formulation

Monochromatic wave field is considered and the problem is formulated in frequency domain by assuming the incident wave potentials at the different orders as:

$$\Phi_I^{(1)}(\mathbf{x}, t) = \Re\left\{\varphi_I^{(1)}(\mathbf{x})e^{-i\omega t}\right\} \quad , \quad \Phi_I^{(2)}(\mathbf{x}, t) = \Re\left\{\varphi_I^{(2)}(\mathbf{x})e^{-2i\omega t}\right\} \tag{11}$$

Consequently all the other quantities will evolve at the same frequencies and can be represented in the same form. Lower case letters are used to denote the frequency domain quantities. The mean value which also occurs at second order, is neglected.

Once the velocity potentials evaluated, the pressure is calculated from the Bernoulli equation and the hydrodynamic forces are calculated by integrating the pressure over the wetted part of the body. The total external hydrodynamic forces are decomposed into the parts dependent on the body motion (added mass $[\,\mathbf{A}\,]$, damping $[\,\mathbf{B}\,]$ and restoring $[\,\mathbf{C}\,]$) and the pure excitation part $\{\mathbf{F}^E\}$, while the internal hydrodynamic forces are decomposed into the added mass $[\,\mathbf{A}^T]$ and the restoring part $[\,\mathbf{C}^T\,]$. Assuming that all the forces are expressed relative to the same reference point, the coupled motion equations become:

$$\{-\omega^2\,([\mathbf{M}] + [\,\mathbf{A}\,(\omega)] + [\,\mathbf{A}^T(\omega)]) - i\omega[\,\mathbf{B}(\omega)\,] + ([\,\mathbf{K}\,] + [\,\mathbf{C}\,] + [\,\mathbf{C}^T\,])\}\left\{\boldsymbol{\xi}^{(1)}\right\} = \left\{\mathbf{F}^{E(1)}\right\} \tag{12}$$

$$\{-4\omega^2\,([\mathbf{M}] + [\,\mathbf{A}\,(2\omega)] + [\,\mathbf{A}^T(2\omega)]) - 2i\omega[\,\mathbf{B}(2\omega)\,] + ([\,\mathbf{K}\,] + [\,\mathbf{C}\,] + [\,\mathbf{C}^T\,])\}\left\{\boldsymbol{\xi}^{(2)}\right\} = \left\{\mathbf{F}^{E(2)}\right\} \tag{13}$$

where $[\mathbf{M}]$ is the modal mass matrix of the structure and $[\,\mathbf{K}\,]$ is the modal structural stiffness matrix.

The solution of these motion equations gives the amplitudes of the body deformations and the problem is formally solved.

Solution for the velocity potentials

The semi-analytical solution for the exterior problem was presented in [2] both at first and second order and will not be repeated. Here we give few more details about the solution of the interior problem. The basic principles are taken from [1] where the modal decomposition method was proposed. The total velocity potential ψ_{Rj} is decomposed into two parts:

$$\psi_{Rj} = \psi_j + \Omega_j \tag{14}$$

The so called Stoke Joukowski velocity potential Ω_j is chosen to satisfy the original body boundary condition at the tank boundaries S_T and the condition at the free surface $(z = h - H)$ as follows:

$$\left.\frac{\partial \Omega_j}{\partial n}\right|_{S_T} = \mathbf{h}_j\mathbf{n} \quad , \quad \left.\frac{\partial \Omega_j}{\partial z}\right|_{z=h-H} = \zeta_j^A \tag{15}$$

It follows that the potential ψ_j satisfy the following body boundary condition and the free surface condition:

$$\left.\frac{\partial \psi_j}{\partial n}\right|_{S_T} = 0 \quad , \quad -\frac{\omega^2}{g}\psi_j + \frac{\partial \psi_j}{\partial z} = \frac{\omega^2}{g}\Omega_j \tag{16}$$

In the present case of the vertical circular cylinder with the beam representation of the structure, the nonhomogeneous term at the free surface ζ_j^A is zero [see (24)] and the solution for the Stoke Joukowski potential can be found by eigenfunction expansion in the following form:

$$\Omega_j(r,\theta,z) = \left[\gamma_{0j}r + \sum_{l=1}^{\infty}\gamma_{lj}I_1(\lambda_l r)\cos\lambda_l(z+H)\right]\cos\theta \tag{17}$$

with $\lambda_l = l\pi/h$ and I_1 denotes the modified Bessel function.
The functions $\cos\lambda_l(z+H)|_{l=0,\infty}$ represent a complete orthogonal basis over the interval $z \in [-H, h-H]$ so that the coefficients γ_{lj} follows in the following form:

$$\gamma_{lj} = \frac{\int_{-H}^{h-H} h_j(z)\cos\lambda_l(z+H)dz}{\int_{-H}^{h-H}\cos^2\lambda_l(z+H)dz} \tag{18}$$

Knowing the Stoke Joukowski velocity potential Ω_j, the complementary velocity potential ψ_j can be expressed as the sum of natural sloshing modes of the interior problem [3]:

$$\psi_j(r,\theta,z) = \sum_{n=1}^{\infty}\alpha_{nj}\frac{\cosh\mu_n(z+H)}{\cosh\mu_n h}J_1(\mu_n r)\cos\theta \tag{19}$$

where $\mu_n = \kappa_n/R$ and κ_n is the n^{th} zero of $\partial J_1(\kappa_n r)/\partial\kappa_n$.
The unknown coefficients α_{nj} are found from the free surface condition (16) which can be written as:

$$\sum_{n=1}^{\infty}\alpha_{nj}(\omega_n^2-\omega^2)J_1(\mu_n r) = \omega^2\left[\gamma_{0j}r + \sum_{l=1}^{\infty}\gamma_{lj}I_1(\lambda_l r)\cos\lambda_l h\right] \tag{20}$$

where $\omega_n^2 = g\mu_n\tanh\mu_n h$ and J_1 denotes the Bessel function.
Since the set of functions $J_1(\mu_n r)|_{n=1,\infty}$ defines a complete orthogonal basis over the interval $r \in [0, R]$, the coefficients α_{nj} follow in the form:

$$\alpha_{nj} = \frac{\omega^2}{\omega_n^2-\omega^2}\left[\gamma_{0j}\frac{\int_0^R rJ_1(\mu_n r)dr}{\int_0^R rJ_1(\mu_n r)J_1(\mu_n r)dr} + \sum_{l=1}^{\infty}\gamma_{lj}\frac{\int_0^R rI_1(\lambda_l r)J_1(\mu_n r)dr}{\int_0^R rJ_1(\mu_n r)J_1(\mu_n r)dr}\cos\lambda_l h\right] \tag{21}$$

Structural modelling

As indicated in the introduction, the structural model is the simplified Euler Bernoulli beam model. This approximation of simple bending is of course questionable for the general case. However, since the goal of the analysis is to validate the overall coupling procedure, we believe that this simple structural modelling validates all the critical steps of the proposed analysis. Within the Euler Bernoulli beam approximation for the beam clamped at one end, the deformation modes can be chosen in many different ways among which the analytical dry modes and Jacobi polynomials are most often used. Here we use the Jacobi polynomials [5] described by the following expression:

$$f_j(z) = q^2P_{j-1}(q) \quad , \quad q = \frac{z+H}{L} \quad , \quad P_n(q) = \sum_{m=0}^{n}(-1)^m\frac{(4+2n-m)!}{m!\,(n-m)!\,(4+n-m)!}q^{n-m} \tag{22}$$

The modal deformation of one point at the body becomes:

$$\mathbf{h}_j(x,y,z) = h_{jx}\mathbf{i} + h_{jy}\mathbf{j} + h_{jz}\mathbf{k} = f_j(z)\mathbf{i} + 0\mathbf{j} - \frac{\partial f_j(z)}{\partial z}x\mathbf{k} \tag{23}$$

It is now easy to show that the term ζ_j^A becomes zero in this case:

$$\zeta_j^A = \frac{\iint_{S_T} \mathbf{h}_j \mathbf{n} dS}{A_w} = \frac{\int_{-H}^{h-H} f_j(z)dz \int_0^{2\pi} \cos\theta \, d\theta}{A_w} = 0 \tag{24}$$

It should be noted that for the general case of tank deformations the value of ζ_j^A is not necessarily zero.

Numerical results

The following tank characteristics have been chosen for the numerical example: $R = 30\text{m}$, $H = L = 100\text{m}$, $h = 75\text{m}$ (filling ratio 0.75), uniformly distributed mass along the length of the cylinder is the one third of the displaced mass, a concentrated mass at the top of the cylinder m_0 (free surface level) is equal to the total displaced mass, the stiffness of the cylinder is chosen such that the ratio EI/L^3 is equal to $0.27m_0s^{-2}$. With those characteristics the first natural sloshing frequency is around 0.77 rad/s and the first bending mode resonance is around 0.84 rad/s. The results for the displacement of the top of the cylinder are shown in Figure 2.

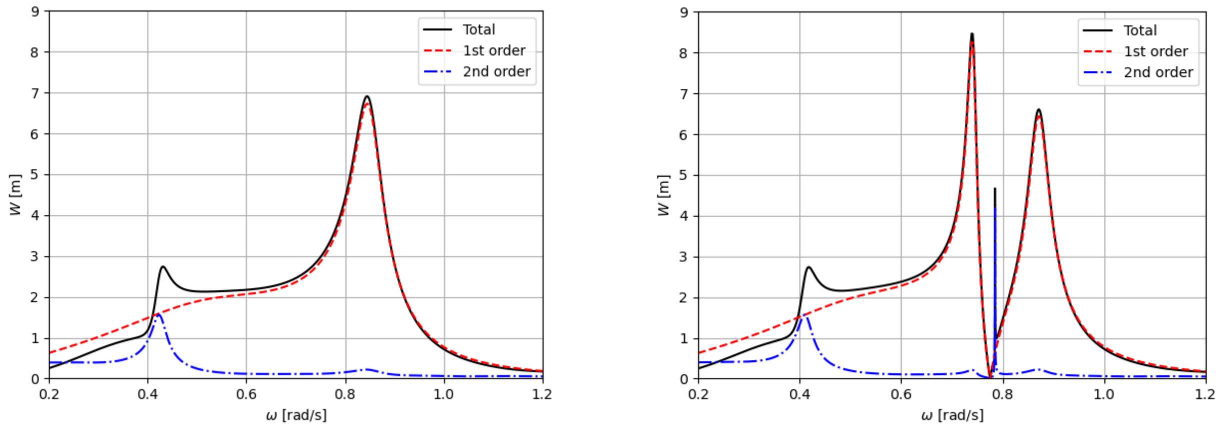

Figure 2: Total displacement of the top of the tank in regular waves of the amplitude $A = 10m$. (Left – without the liquid in the tank, Right – with liquid in the tank).

The amplitude of the incident wave is set to be 10m and the total displacement of the top of the column is presented. The influence of sloshing can be clearly observed close to the 1st sloshing resonance frequency. A typical two peak response is induced by the dynamic interactions of the internal and external fluid flow. One additional peak in the response, around the 1st sloshing resonance frequency, can also be observed and it is due to the excitation of the 1st sloshing mode induced by the second order vibrations of the column.

Discussions

We have presented here the general methodology for the analytical solution of the water wave interactions with partially filled circular cylindrical tank fixed at the sea bottom. The characteristic influence of sloshing on global body behavior was reproduced at first and second orders. It should be noted that the similar problem (only linear!) was treated in [1] using a slightly simplified approach based on strip theory which was possible thanks to the considered operating conditions. The present approach is fully consistent 3D approach and can be applied to any particular cylinder configurations. Due to the high precision of the semi-analytical approach, the results can be used for the validation of numerical results. Applying the same methodology to the membrane type of structural modelling is left for further work.

References

[1] Faltinsen O.M. & Timokha A.N., 2009. : "*Sloshing*." Cambridge University Press.
[2] Choi Y.M. & Malenica S., 2013: "*Semi analytical solution for second order hydroelastic response of the vertical circular cylinder in monochromatic water waves*.", 28th IWWWFB, L'isle sur la Sorgue, France
[3] Ouled Housseine C., 2015. : "*Semi-analytical solution for coupling between sloshing and wave-induced vibrations of liquid filled vertical circular cylinder*.", Unpublished note.
[4] Malenica S, Bigot F., Chen X.B. & Bralic S., 2011. : "*Global hydroelastic model for LNG ships*.", 26th IWWWFB Athens, Greece.
[5] Newman J.N., 1994. : "*Wave effects on deformable bodies*.", Applied Ocean Research, Vol. 16.
[6] Strand I.M. & Faltinsen O.M., 2019. : "*Linear wave response of a 2D closed flexible fish cage*.", Journal of Fluids and Structures, Vol 87., pp. 58-83.