Abstract for 36th IWWWFB, Seoul National University, Korea - April 25-28, 2021

Radiation of water waves by a heaving disc in a uniform current

Hui Liang[1], Xiaobo Chen[2,3], Eng Soon Chan[1]

1: Technology Centre for Offshore and Marine, Singapore (TCOMS), 118411, Singapore
2: Research Department, Bureau Veritas, 8 Cours du Triangle, 92937, Paris La Defense, France
3: College of Shipbuilding Engineering, Harbin Engineering University, Harbin, 150001, China
E-mail: liang_hui@tcoms.sg

1 Highlights

- A Galerkin-type hypersingular integral equation taking the translating and pulsating source function as the fundamental solution is developed to study the wave radiation problem by a heaving submerged disc.
- The potential jump across the disc is represented by the Fourier-Gegenbauer series which incorporate the square-root singularity at the edge.
- At the critical frequency, both wave force and the wave amplitude are finite, and planar resonant waves are generated.

2 Basic equations

Consider a circular disc of a radius L and zero thickness undergoing heaving motion is submerged beneath a free surface in deep water. It is assumed that the fluid is inviscid and incompressible, and flow is irrotational so that a velocity potential exists. For the sake of simplicity, all physical variables in the present study are nondimensionalised. Nondimensional coordinates $\boldsymbol{x} = (x, y, z)$, time t, and velocity potential Φ are with respect to L, $\sqrt{L/g}$, and $\sqrt{gL^3}$, respectively, where g denotes the gravitational acceleration. A 3D coordinate system $Oxyz$ is chosen fixed to the mean position of the disc, with the Oxy plane coinciding with the undisturbed free surface and Oz axis pointing positively upward through the centre of the disc. The disc has a non-dimensional submergence d. In addition, we define nondimensional angular frequency $f = \omega\sqrt{L/g}$, Froude number $F = U/\sqrt{gL}$, and their product Brard number $\tau = Ff = U\omega/g$, where ω and U denote dimensional heaving frequency and current velocity, respectively. The uniform current is along the negative x-direction.

The total velocity potential in the flow field can be decomposed into a steady-flow potential $-Fx$ and an unsteady radiation potential:

$$\Phi(\boldsymbol{x}, t) = -Fx + \mathrm{Re}\left[-\mathrm{i}fX_3\phi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}ft}\right], \tag{1}$$

where X_3 denotes the amplitude of the heaving motion, and ϕ means the corresponding radiation potential. On the free surface, the linearised free-surface boundary condition in the frequency domain is:

$$-f^2\phi + 2\mathrm{i}\tau\frac{\partial\phi}{\partial x} + F^2\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = 0. \tag{2}$$

On the disc surface $\mathscr{D}$, the normal velocity is prescribed and is determined by the body boundary condition

$$\left.\frac{\partial\phi}{\partial n}\right|_{\mathscr{D}} = 1. \tag{3}$$

where n is the vector normal to the disc surface which is defined positive pointing upward. In the body boundary (3), there is no contribution from m_j-terms in the present problem.

A disc of zero thickness can be represented by a dipole distribution, and then the boundary integral equation is written as

$$\phi(\boldsymbol{x}) = -\frac{1}{4\pi}\iint_{\mathscr{D}}\psi(\boldsymbol{\xi})\frac{\partial G(\boldsymbol{x};\boldsymbol{\xi})}{\partial n_{\boldsymbol{\xi}}}\mathrm{d}S, \tag{4}$$

where $\boldsymbol{x} \equiv (x, y, z)$ and $\boldsymbol{\xi} \equiv (\xi, \eta, \zeta)$ denote the flow-field point and singularity point, respectively. $\psi(\boldsymbol{\xi}) = \phi^+ - \phi^-$ means the velocity potential jump across the disc, where $\pm$ denote the upper and lower surfaces of the disc, respectively. In addition, G is the translating and pulsating source Green function expressed as:

$$G = G^R + G^F \quad \text{with} \quad G^R = -[R^2 + (z-\zeta)^2]^{-1/2} + [R^2 + (z+\zeta)^2]^{-1/2} \quad \text{and} \quad R = \sqrt{(x-\xi)^2 + (y-\eta)^2}, \tag{5}$$

where the free-surface term G^F is written as [1]

$$G^F = \frac{1}{\pi}\lim_{\epsilon\to 0+}\int_{-\pi}^{\pi}\int_0^{\infty}\frac{\kappa}{D + \mathrm{i}\epsilon D_f}\mathrm{e}^{\kappa(z+\zeta) - \mathrm{i}\kappa[(x-\xi)\cos\theta + (y-\eta)\sin\theta]}\mathrm{d}\kappa\mathrm{d}\theta \quad \text{with} \quad D = (F\kappa\cos\theta - f)^2 - \kappa, \tag{6}$$

where ϵ is introduced to satisfy the radiation condition in the far field. To determine the potential jump $\psi(\boldsymbol{\xi})$, the body boundary condition (3) is recalled, and a hypersingular integral equation is constructed on the disc surface:

$$-\frac{1}{4\pi}\iint_{\mathscr{D}}\psi(\boldsymbol{\xi})\frac{\partial^2 G(\boldsymbol{x};\boldsymbol{\xi})}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{\xi}}}\mathrm{d}S = \frac{\partial\phi(\boldsymbol{x})}{\partial n_{\boldsymbol{x}}}. \tag{7}$$

To evaluate the translating and pulsating source Green function, we adopt the formulae in [2] which is suited to integration over a smooth surface. We define the polar coordinates: $(x, y) = h(\cos\gamma, \sin\gamma)$ and $(\xi, \eta) = \varrho(\cos\varphi, \sin\varphi)$. By applying the Jacobi-Anger expansion [3], the free-surface term in the Green function is written as [2]:

$$G^F = \frac{1}{\pi}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \mathrm{i}^{-p+q}\mathrm{e}^{\mathrm{i}(p\gamma+q\varphi)}\int_0^\infty \mathrm{e}^{\kappa(z+\zeta)}g_{p+q}(\kappa)J_p(\kappa h)J_q(\kappa\varrho)\mathrm{d}\kappa \quad \text{with} \quad g_\ell(\kappa) = \lim_{\epsilon\to 0+}\int_{-\pi}^{\pi}\frac{\kappa\mathrm{e}^{\mathrm{i}\ell\theta}}{D+\mathrm{i}\epsilon D_f}\mathrm{d}\theta, \tag{8}$$

In Eq. (8), $g_\ell(\kappa)$ can be expressed analytically, and elaborate analysis has been presented in [2].

3 Galerkin method for the solution of the integral equation

To accurately resolve the singular behaviours of fluid velocities at the edge, a Galerkin method is applied to solve the hypersingular boundary integral equation (7). At the edge of the circular disc, the velocity potential jump is zero whereas the velocity components are singular

$$\psi(\varrho,\varphi) = 0 \quad \text{and} \quad \|\nabla\psi(\varrho,\varphi)\| \sim (1-\varrho)^{-1/2} \quad \text{at} \quad \varrho = 1. \tag{9}$$

To incorporate the null potential jump condition and the square-root singular behaviour of velocity components in Eq. (9) at the edge of the disc, the velocity jump ψ is expanded into a Fourier-Gegenbauer series [4]:

$$\psi(\varrho,\varphi) = \sum_{k=0}^{\infty}\sum_{l=-\infty}^{\infty}\psi_{k,l}\Psi_k^{|l|}(\varrho)\mathrm{e}^{\mathrm{i}l\varphi} \quad \text{with} \quad \Psi_k^{|l|}(\varrho) = \frac{k!\Gamma(|l|+\frac{1}{2})}{\sqrt{2\pi}\Gamma(k+|l|+\frac{3}{2})}\varrho^{|l|}C_{2k+1}^{|l|+1/2}(\sqrt{1-\varrho^2}), \tag{10}$$

where $C_a^b(u)$ is the Gegenbauer polynomial [3]. The radial base function $\Psi_k^{|l|}(\varrho)$ satisfies the following orthogonal relation [5]:

$$\int_0^1 \frac{x\Psi_k^{|l|}(x)\Psi_m^{|l|}(x)}{\sqrt{1-x^2}}\mathrm{d}x = \begin{cases} 0 & m \neq k, \\ \dfrac{(2m+2|l|+1)!(m!)^2}{4^{|l|+1}(2m+1)!(2m+|l|+\frac{3}{2})[\Gamma(m+|l|+\frac{3}{2})]^2} & m = k. \end{cases} \tag{11}$$

Substituting Eq. (10) into the hypersingular integral equation (7) and applying the Galerkin collocation via integrating a test function $h\Psi_m^{|n|}(h)\mathrm{e}^{-\mathrm{i}n\gamma}$ over the disc area in the sense of [4, 5, 6] gives rise to a linear equation system $[\mathfrak{A}]\cdot\{\boldsymbol{\psi}\} = \{\boldsymbol{b}\}$. The elements in matrix $[\mathfrak{A}]$ are expressed as

$$\mathfrak{A}_{m,n;k,l} = \mathfrak{A}^R_{m,n;k,l} + \mathfrak{A}^F_{m,n;k,l}, \tag{12}$$

where $\mathfrak{A}^R_{m,n;k,l}$ and $\mathfrak{A}^F_{m,n;k,l}$ are written as

$$\mathfrak{A}^R_{m,n;k,l} = \pi\delta_{n,l}\left[-\frac{\delta_{m,k}}{4m+2|n|+3} - \int_0^\infty \kappa^{-1}\mathrm{e}^{-2\kappa d}J_{|n|+2k+3/2}(\kappa)J_{|n|+2m+3/2}(\kappa)\mathrm{d}\kappa\right], \tag{13}$$

$$\mathfrak{A}^F_{m,n;k,l} = -\mathrm{i}^{-n+l}\chi_l\chi_n\int_0^\infty \frac{\mathrm{e}^{-2\kappa d}}{\kappa}g_{n-l}(\kappa)J_{|l|+2k+3/2}(\kappa)J_{|n|+2m+3/2}(\kappa)\mathrm{d}\kappa \quad \text{with} \quad \chi_l = \begin{cases} 1 & \text{when} \quad 0 \le l \\ (-1)^l & \text{otherwise} \end{cases} \tag{14}$$

and elements in the vector $\{\boldsymbol{b}\}$ are written as

$$b_{m,n} = \int_0^1\int_{-\pi}^{\pi}\frac{\partial\phi(\boldsymbol{x})}{\partial n_{\boldsymbol{x}}}h\Psi_m^{|n|}(h)\mathrm{e}^{-\mathrm{i}n\gamma}\mathrm{d}\gamma\mathrm{d}h = \frac{2\sqrt{2\pi}}{3}\delta_{n,0}\delta_{m,0}. \tag{15}$$

4 Hydrodynamic forces and wave pattern

By solving the linear equation system, the coefficients $\psi_{k,l}$ can be determined. Based on the Bernoulli's equation, the nondimensional vertical force is

$$F_3 = \int_0^1\int_{-\pi}^{\pi}\left[-\mathrm{i}f\psi - F\left(\frac{\partial\psi}{\partial h}\cos\gamma - \frac{\partial\psi}{h\partial\gamma}\sin\gamma\right)\right]h\mathrm{d}\gamma\mathrm{d}h. \tag{16}$$

According to the identity $\frac{\mathrm{d}}{\mathrm{d}u}C_n^{(\lambda)}(u) = 2\lambda C_{n-1}^{(\lambda+1)}(u)$, the integral of the advection term in (16) is null, and therefore the vertical force is simplified to

$$F_3 = -\mathrm{i}f\psi_{0,0}\frac{2\sqrt{2\pi}}{3}. \tag{17}$$

Based on the integral equation (4) and the dynamic free-surface boundary condition, the free-surface elevation $\mathcal{E}$ is expressed as

$$\mathcal{E} = \mathrm{i}f\phi + F\frac{\partial\phi}{\partial x} = -\frac{1}{4\pi}\iint_{\mathscr{D}}\psi(\boldsymbol{\xi})\left[\mathrm{i}f\frac{\partial G(\boldsymbol{x},\boldsymbol{\xi})}{\partial n_{\boldsymbol{\xi}}} + F\frac{\partial^2 G(\boldsymbol{x};\boldsymbol{\xi})}{\partial x\partial n_{\boldsymbol{\xi}}}\right]\mathrm{d}S. \tag{18}$$

Substituting the Fourier-Gegenbauer series (10) for ψ and the expansion for the free-surface Green function given by (8) into expression (19) yields

$$\mathcal{E}=\sum_{k=0}^{\infty}\sum_{l=-\infty}^{\infty}\frac{-\psi_{k,l}}{2\pi}\sum_{p=-\infty}^{\infty}\mathrm{i}^{l-p}\chi_l\mathrm{e}^{\mathrm{i}p\gamma}\int_0^{\infty}\frac{\mathrm{e}^{-\kappa d}}{\sqrt{\kappa}}g_{p-l}(\kappa)\left[\left(\mathrm{i}f-\frac{\mathrm{i}pF}{h}\sin\gamma\right)J_p(\kappa h)+F\kappa J_p'(\kappa h)\cos\gamma\right]J_{|l|+2k+3/2}(\kappa)\mathrm{d}\kappa. \tag{19}$$

5 Critical frequency

The wave forces on a circular disc at the critical frequency associated with $\tau=Ff=1/4$ are now considered. Although the Green function at the critical frequency is unbounded, yet Grue & Palm [7] provided strong evidence that the generated wave amplitude is finite. Then, Liu & Yue [8] made a rigorous proof using a source distribution to confirm that the hydrodynamic forces are bounded as long as the displaced volume is nonzero. In the present problem, however, the disc of zero water displacement is represented by a dipole distribution. For this reason, this study is beyond the scope of application of the study by Liu & Yue [8], and the generalisation of their study is considered here.

When τ is approaching 1/4, the wavenumber integral (14) is dominated by the integral over an interval in the neigbourhood of $\kappa_0=1/(4F^2)$. By performing the asymptotic analysis to $g_\ell(\kappa)$, the asymptotic expression of the Green function is expressed as [9]:

$$G=4\sqrt{2}\mathrm{i}\kappa_0\mathrm{e}^{\kappa_0[(z+\zeta)+\mathrm{i}(x-\xi)]}\log\delta+\tilde{G}\quad\text{with}\quad\delta^2=|1-4\tau|, \tag{20}$$

in which the component $\tilde{G}$ is of the order of $O(1)$. Substituting (20) into the integral equation (7) yields

$$-\frac{\sqrt{2}\mathrm{i}}{\pi}\kappa_0^3\mathrm{e}^{\kappa_0(z+\mathrm{i}x)}\log\delta\iint_S\psi(\boldsymbol{\xi})\mathrm{e}^{\kappa_0(\zeta-\mathrm{i}\xi)}\mathrm{d}S(\boldsymbol{\xi})-\frac{1}{4\pi}\iint_S\psi(\boldsymbol{\xi})\frac{\partial^2\tilde{G}(\boldsymbol{x};\boldsymbol{\xi})}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{\xi}}}\mathrm{d}S(\boldsymbol{\xi})=\frac{\partial\phi(\boldsymbol{x})}{\partial n_{\boldsymbol{x}}}. \tag{21}$$

Define the Kochin function as

$$\mathcal{K}=\iint_S\psi(\boldsymbol{x})\mathrm{e}^{\kappa_0(z-\mathrm{i}x)}\mathrm{d}S(\boldsymbol{x}). \tag{22}$$

From (21), the Kochin function is expressed as

$$\mathcal{K}=\frac{\pi}{\sqrt{2}\mathrm{i}\kappa_0^3\Gamma}\left[-\mathcal{F}-\frac{1}{4\pi}\iint_S\psi(\boldsymbol{\xi})P(\boldsymbol{\xi})\mathrm{d}S(\boldsymbol{\xi})\right]\log^{-1}\delta, \tag{23}$$

with Γ, $\mathcal{F}$ and $P(\boldsymbol{\xi})$ defined as

$$\Gamma=\iint_S\mathrm{e}^{2\kappa_0z}\mathrm{d}S(\boldsymbol{x}),\quad\mathcal{F}=\iint_S\frac{\partial\phi(\boldsymbol{x})}{\partial n_{\boldsymbol{x}}}\mathrm{e}^{\kappa_0(z-\mathrm{i}x)}\mathrm{d}S(\boldsymbol{x}),\quad\text{and}\quad P(\boldsymbol{\xi})=\iint_S\mathrm{e}^{\kappa(z-\mathrm{i}x)}\frac{\partial^2\tilde{G}(\boldsymbol{x};\boldsymbol{\xi})}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{\xi}}}\mathrm{d}S(\boldsymbol{x}). \tag{24}$$

Equation (23) indicates that the Kochin function is of the order of $O(\log^{-1}\delta)$ as long as the disc is of finite area ($\Gamma\neq0$). Substituting the Kochin function $\mathcal{K}$ in (23) into the integral equation (21) and integrating a function $\mathrm{e}^{\kappa_0(z-\mathrm{i}x)}$ over the disc surface yields

$$\iint_S\psi(\boldsymbol{\xi})\left[\mathrm{e}^{\kappa_0(z+\mathrm{i}x)}P(\boldsymbol{\xi})-\Gamma\frac{\partial^2\tilde{G}(\boldsymbol{x};\boldsymbol{\xi})}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{\xi}}}\right]\mathrm{d}S(\boldsymbol{\xi})=4\pi[\Gamma-\mathcal{F}\mathrm{e}^{\kappa_0(z+\mathrm{i}x)}]. \tag{25}$$

In (25), the right-hand side term is of the order of $O(1)$, so is the kernel of of the integral equation. Therefore, it is inferred that the potential jump ψ has the same order, i.e.: $O(1)$. Following the similar procedure, the velocity potential is

$$\phi(\boldsymbol{x})=\frac{\mathrm{e}^{\kappa_0(z+\mathrm{i}x)}}{\kappa_0\Gamma}\left[\mathcal{F}+\frac{1}{4\pi}\iint_S\psi(\boldsymbol{\xi})P(\boldsymbol{\xi})\mathrm{d}S(\boldsymbol{\xi})\right]-\frac{1}{4\pi}\iint_S\psi(\boldsymbol{\xi})\frac{\partial\tilde{G}(\boldsymbol{x};\boldsymbol{\xi})}{\partial n_{\boldsymbol{\xi}}}\mathrm{d}S(\boldsymbol{\xi}), \tag{26}$$

which indicates that the potential ϕ has an order of $O(1)$. By accounting for the Fourier-Gegenbauer series (10) and expression (19), the free-surface elevation at $\tau=1/4$ is expressed as

$$\mathcal{E}=4\sqrt{2}\kappa_0(\tau+\kappa_0)\exp[\kappa_0(-d+\mathrm{i}x)]\sum_{k=0}^{\infty}\sum_{l=-\infty}^{\infty}\psi_{k,l}\chi_l(-\mathrm{i})^lJ_{|l|+2k+3/2}(\kappa_0), \tag{27}$$

which indicates that the generated waves are also bounded. Moreover, the spatial term $\exp[\kappa_0(-d+\mathrm{i}x)]$ means that a heaving submerged disc generates plane waves along x-axis at the critical frequency $\tau=1/4$. The occurrence planar waves is due to resonance. According to Li & Liu [10], the resonant waves will become evanescent when the third-order free-surface nonlinearity is accounted for.

6 Results and discussions

Figs. 1 and 2 depict the nondimensional heave added mass and wave-radiation damping coefficients versus the nondimensional frequency f at a submergence $d = 0.5$ for Froude numbers $F = 0.2$ and $F = 0.5$, respectively. The middle and right-hand side subplots display the variation of added mass and damping coefficients in the vicinity of the critical frequency $\tau = 1/4$.

At $F = 0.2$ as in Fig. 1, the variation across the critical frequency is not apparent, and the added mass and damping coefficients are approaching same values near $\tau = 1/4$. In contrast, a sharp change in proximity to $\tau = 1/4$ is witnessed at $F = 0.5$ as in Fig. 2. The middle and right subplots of Fig. 2 show that both added mass and wave-radiation damping drop dramatically when τ is approaching $1/4$. With the vanishing of $\tau - 1/4$, however, the variation of hydrodynamic forces becomes mild, and the added mass and wave-radiation damping coefficients tend to approach same results which is consistent with the conclusion we have drawn.

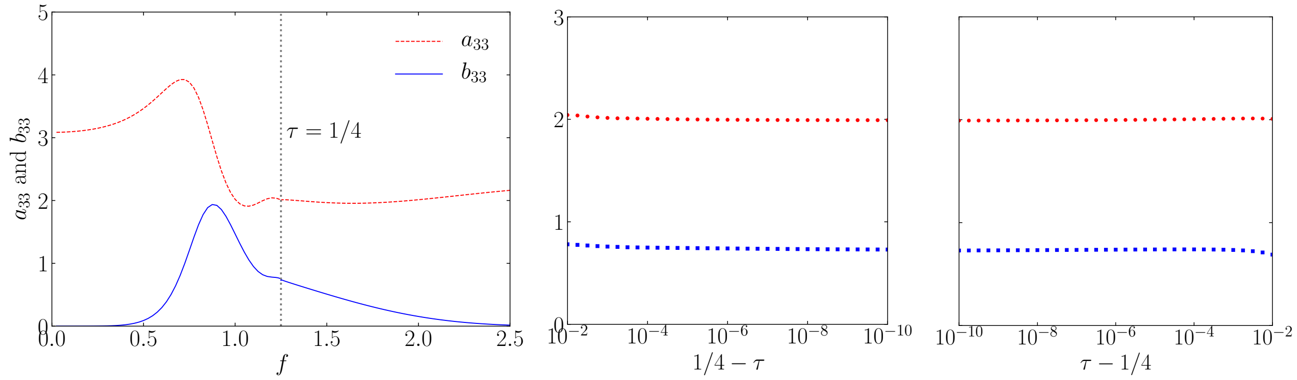

Figure 1: Added mass and wave-radiation damping coefficients at $F = 0.2$.

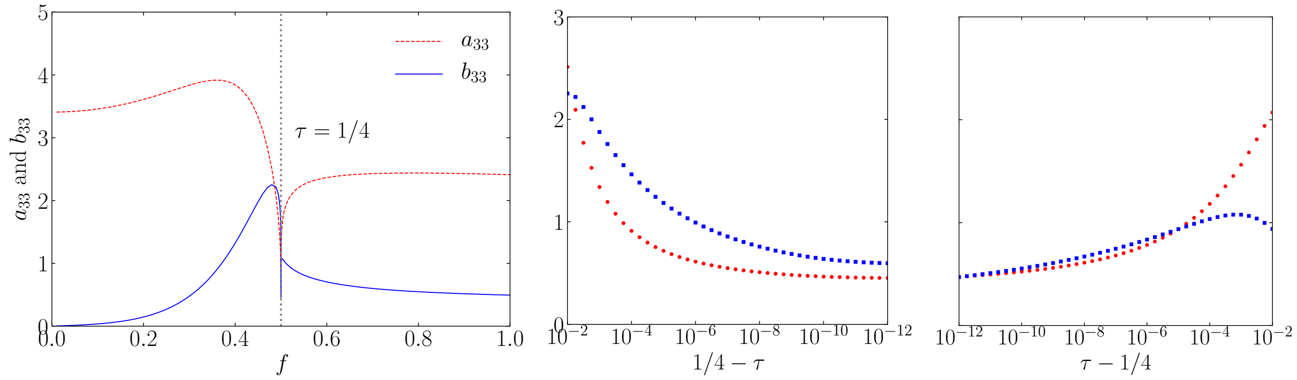

Figure 2: Added mass and wave-radiation damping coefficients at $F = 0.5$.

Acknowledgment: The first author is grateful to Dr. Yuming Liu at MIT for many fruitful discussions.

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