Wet and dry modes of complex structures in impact problems

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1. Introduction. The present study is concerned with dry and wet modes of an elastic structure in either full or partial contact with a liquid. The problem is two-dimensional, linear, without any damping and external forcing. Initial conditions are not required in this problem. The liquid is of infinite depth with horizontal equilibrium level. The shape of the structure, without vibrations and without bending stresses in it, can be approximated as flat and close to the equilibrium level of the liquid in the leading order. Such cases include, in particular, floating elastic plates of small thickness and elastic wedges with small deadrise angles during the early stage of their impact onto the water surface. The structural deflection is described by a one-dimensional equation of thin elastic plate. The thickness of the plate and the elastic properties of the plate vary, in general, along the plate. The plate is of finite length with certain edge conditions. There could be some extra supports at internal points of the plate. The lower surface of the plate can be either completely wetted or in a partial contact with the liquid.

The assumption of no damping implies that gravity, surface tension and viscous effects, as well as the structural damping, are not included into the model. This assumption is justified for structures with relatively high frequencies of its natural vibrations. The assumptions of small thickness of the plate and linear response of the plate mean that both the plate thickness and the plate deflection are much smaller than the plate length. As a result, the boundary conditions on the liquid free surface and on the liquid/structure interface can be linearised and imposed on the equilibrium liquid surface.

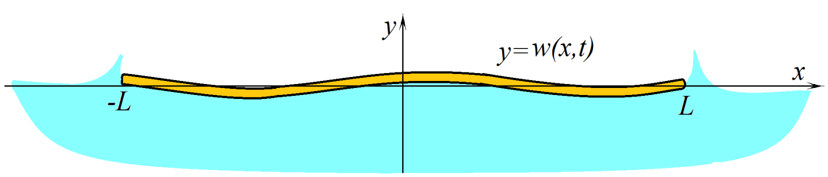

Figure 1. Floating elastic plate and notation.

The problem is coupled. The frequencies and the corresponding shapes of the structure vibrations, which are known as the wet modes, should be determined together with the hydrodynamic pressures caused by these vibrations. The wet modes are sought as superposition of dry modes of the same structure. The dry modes and their dry natural frequencies are assumed known for a given structure. This approach gives rise to a so-called added-mass matrix, which describes interactions of the dry modes through the contact with the liquid. The dry modes are independent and orthogonal if the structure is not in contact with the liquid.

We shall explain how to calculate the added-mass matrix for free-free floating elastic plate of constant thickness, and how to calculate the wet modes and corresponding wet frequencies. It will be explained how to apply the developed algorithm to (1) other edge conditions, (2) to plates in partial contact with liquid, (3) to plates with internal supports and non-constant thickness, (4) to several plates in full or partial contact with liquid, and (5) to plates with rigid parts of non-zero length.

2. Formulation of coupled problems for a floating plate. The deflection of a free-free floating plate $w(x,t)$ is described by the equations,

$$mw_{tt} + (Dw'')'' = -\rho\varphi_t \qquad (-L < x < L), \qquad w'' = w''' = 0 \qquad (x = \pm L), \tag{1}$$

$$\nabla^2\varphi = 0 \qquad (y < 0), \qquad \varphi = 0 \qquad (y = 0,\ |x| > L), \qquad \varphi_y = w_t \qquad (y = 0,\ |x| < L), \tag{2}$$

where m is the mass of the plate per unit area, D is the rigidity coefficient, ρ is the liquid density. A prime stands for x-derivative. The velocity potential $\varphi(x, y, t)$ satisfies Laplace's equation in the flow region $y < 0$, linearised dynamic condition on the free surface, $y = 0$ $|x| > L$, the linearised body condition in the wetted part of the plate, $y = 0$ $|x| < L$, and decays at infinity, $\varphi \to 0$ as $x^2 + y^2 \to \infty$. We shall determine periodic in time non-trivial solutions of coupled problem (1), (2) in the form

$$w(x,t) = W(x)\cos(\omega t), \qquad \varphi(x,y,t) = \Phi(x,y)\sin(\omega t), \tag{3}$$

where $W(x)$ and $\Phi(x, y)$ are real-valued functions to be determined together with the natural frequency ω. The problem with respect to the shape of the plate natural vibration, $W(x)$, and the corresponding potential, $\Phi(x, y)$, is formulated in the dimensionless variables, which are denoted with tilde,

$$x = L\tilde{x}, \quad W = W_{sc}\tilde{W}, \quad \Phi = \omega L W_{sc}\tilde{\Phi}, \qquad \Omega = \frac{\rho L^5}{D}\omega^2, \qquad \alpha = \frac{m}{\rho L},$$

where W_{sc} is a formal scale of plate deflection. The tilde is dropped below. All variables and parameters are dimensionless in the following analysis. Equations (1) and (2) provide

$$W^{iv} = \Omega\Big[\alpha W - \Phi(x,0)\Big] \qquad (|x| < 1), \qquad W'' = W''' = 0 \qquad (x = \pm 1), \tag{4}$$

$$\nabla^2\Phi = 0 \qquad (y < 0), \qquad \Phi = 0 \qquad (y = 0,\ |x| > 1), \qquad \Phi_y = -W \qquad (y = 0,\ |x| < 1). \tag{5}$$

For a plate with variable thickness and/or elasticity, m and D in (1) are functions of x and the equation in (4) should be modified. For different edge supports, the edge conditions in (4) should be changed. The presence of internal supports brings more conditions to (4), however, all these modifications do not change significantly the formulation of the wet-mode problem. If the plate is in partial contact with liquid, then the interval, where the body boundary condition (5) is imposed, should be adjusted.

The problem (4), (5) and its meaningful modifications have discrete number of non-zero solutions $W_k(x)$ and the corresponding eigenvalues Ω_k, where $k \geq 1$, $\Omega_k < \Omega_{k+1}$ and the wet frequencies of the wet modes are given by $\omega_k = [\Omega_k D/(\rho L^5)]^{\frac{1}{2}}$. The non-zero solutions of (1), where $\Phi(x, 0) = 0$, are known as the dry modes $\psi_n(x)$ with eigenvalues $\lambda_n^4 = \alpha\Omega_n$. The corresponding dry frequencies are $\omega_n^{(d)} = \lambda_n^2[D/(mL^4)]^{\frac{1}{2}}$.

3. Wet modes through the dry modes. The solutions of (1) are sought as superposition of the dry modes,

$$W(x) = \sum_{n=1}^{\infty} W_n\psi_n(x), \tag{6}$$

with coefficients W_n to be determined. Here $\psi_n(x)$, $n \geq 1$, are the solutions of the spectral problem,

$$\psi_n^{(iv)} = \lambda_n^4\psi_n \qquad (|x| < 1), \qquad \psi_n'' = \psi_n''' = 0 \qquad (x = \pm 1), \tag{7}$$

λ_n is a spectral parameter, $n \geq 1$. These non-zero solutions are the dry modes of the plate. There are two modes, $\psi_1(x) = 1/\sqrt{2}$ and $\psi_2(x) = \sqrt{3/2}x$ with $\lambda_1 = \lambda_2 = 0$, for the free-free plate, which correspond to rigid motions of the plate (rigid modes). Numbers of the elastic modes start from $n = 3$. The modes are orthonormal, $n \geq 1$,

$$\int_{-1}^{1} \psi_n(x)\psi_m(x)dx = \delta_{nm}. \tag{8}$$

Equations (5) and (6) provide the corresponding decomposition of the potential and the condition on the plate for the new potentials $\phi_n(x,y)$,

$$\Phi(x,y) = -\sum_{n=1}^{\infty} W_n \phi_n(x,y), \qquad \frac{\partial \phi_n}{\partial y} = \psi_n(x) \quad (|x|<1,\ y=0). \tag{9}$$

Substituting (6), (9) in (4) and using (7) and (8) leads to infinite system of linear algebraic equations for the vector $\vec{W} = (W_1, W_2, W_3, ...)^T$,

$$D\vec{W} = \Omega(\alpha I + S)\vec{W}, \qquad S_{nm} = \int_{-1}^{1} \phi_n(x,0)\psi_m(x)dx, \qquad D_{nm} = 0 \ \ (n \neq m), \quad D_{nn} = \lambda_n^4, \tag{10}$$

where D is the diagonal matrix, I is the unit matrix, and S is the symmetric added-mass matrix. The wet modes ω_k are expressed through Ω_k, which are real positive solutions of the equation $\det[D - \Omega(\alpha I + S)] = 0$. The vector $\vec{W}_k$ of the coefficients of the kth wet mode $W_k(x)$ is obtained as a solution of the truncated system (10) with excluded kth row and $W_{kk} = 1$. We do not normalise the wet modes. The rigid dry and wet modes are equal, $W_1(x) = \psi_1(x)$ and $W_2(x) = \psi_2(x)$, with $\Omega_1 = \Omega_2 = 0$. Difficulties with calculations of the wet modes are due to quick growth of the elements of the matrix D as $n \to \infty$, as well as with evaluation of the elements of the added-mass matrix S_{nm} which are given by double singular integrals. The integrals S_{nm} are calculated analytically in the next section.

If a plate is of non-constant thickness, supported in internal points with different edge conditions, then one should modify the dry-plate equations (7) and (8) accordingly. If the plate is wetted partly, then the limits in the elements of the added-mass matrix, see (10), should be adjusted.

4. Added-mass matrices for simple and complex structures. The elements of the added-mass matrix S_{nm} depend on the dry modes $\psi_n(x)$ and the contact interval. The elements can be presented as a bilinear operator $S_{nm} = U\big[\psi_m(x), \psi_n(x)\big]$. Any solution of equation (7) with any edge conditions and with a real $\lambda_n^4 > 0$ has the form

$$\psi_n(x) = L_{n1}f_{n1}(x) + L_{n2}f_{n2}(x) + L_{n3}f_{n3}(x) + L_{n4}f_{n4}(x),$$

$$f_{n1}(x) = \cos(\lambda_n x), \ \ f_{n2}(x) = \sin(\lambda_n x), \ \ f_{n3}(x) = e^{-\lambda_n(1+x)}, \ \ f_{n4} = e^{-\lambda_n(1-x)},$$

where L_{nj} are coefficients specific for imposed edge conditions. Then, $S_{nm} = \vec{L}_m \cdot S_e^{(nm)} \cdot \vec{L}_n$, where $\vec{L}_n = (L_{n1}, L_{n2}, L_{n3}, L_{n4})$, and $S_e^{(nm)}$ is a symmetric 4×4 matrix with the elements $Se_{ij}^{(nm)} = U\big[f_{mj}(x), f_{nj}(x)\big]$, $1 \leq i,j \leq 4$, $n \geq 3$ and $m \geq 3$. The elements $Se_{ij}^{(nm)}$ are evaluated using the formula

$$U\big[e^{ax}, e^{bx}\big] = \begin{cases} \frac{\pi}{a+b}\big[I_0(a)I_1(b) + I_1(a)I_0(b)\big] & (a \neq -b), \\ \pi\big[I_0^2(a) - I_1^2(a) - \frac{1}{a}I_0(a)I_1(a)\big] & (a = -b), \end{cases} \tag{11}$$

with any complex-valued a and b.

If the plate is wetted partly, only formula (11) should be modified. If a plate is of non-constant thickness with transverse cracks and/or supported in internal points, then the dry modes should be determined and approximated by a series of corresponding dry modes of a homogeneous plate without cracks and internal supports, see Khabakhpasheva et al. (2013) for details of this approach.

5. Numerical results are shown for floating free-free elastic plate of constant thickness in dimensionless variables. The only parameter of the problem, α, is small in the theory of thin plates. The ratios of wet, ω_n, and dry, $\omega_n^{(d)}$, elastic natural frequencies are shown in the left Figure 2 for different numbers $n \geq 3$ and different α. The ratios increase with increase of α, always smaller than one, and approach 1 for large n. Wet frequencies are also defined for $\alpha = 0$, where the ratios tend to zero. Dimensionless wet frequencies for small α are shown in the right Figure 2.

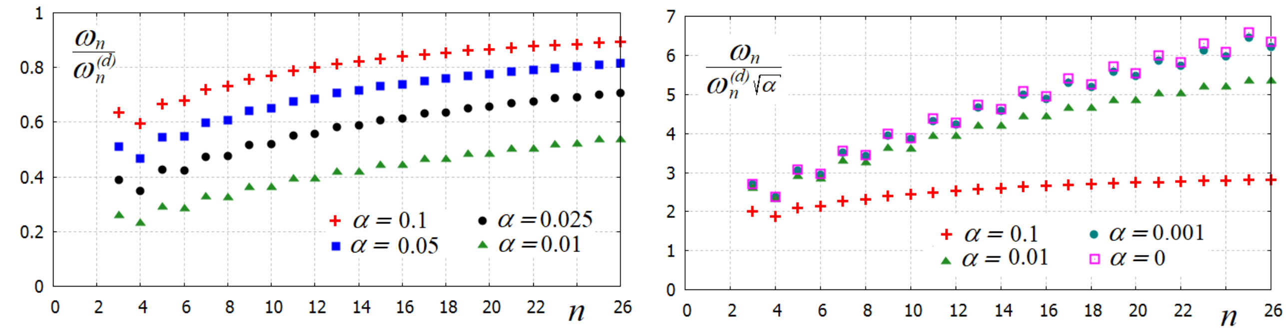

Figure 2. The ratios of wet and dry frequencies (left) and the scaled ratios for small α (right).

The wet modes $W_k(x)$ are obtained as superposition of the dry modes, see (6).

$$W_k(x) = \sum_{n=1}^{\infty} W_{kn}\psi_n(x).$$

The coefficients W_{kn} in such series are shown in the tables below for $\alpha = 0.01$. The left table is for even modes and the right table is for odd modes. It is seen that the wet modes have approximately the same shapes as the corresponding dry modes, but the natural frequencies of the modes are very different, see Figure 2.

$n\backslash k$	3	5	7	9	11	13
1	0.066	0.009	0.003	0.001	0.001	0.000
3	1.0	0.095	0.030	0.013	0.007	0.004
5	-0.098	1.0	0.141	0.060	0.032	0.019
7	-0.017	-0.153	1.0	0.015	0.073	0.042
9	-0.005	-0.042	-0.173	1.0	0.153	0.077
11	-0.002	-0.017	-0.057	-0.181	1.0	0.15
13	-0.001	-0.011	-0.026	-0.065	-0.183	1.0
15	-0.000	-0.008	-0.014	-0.033	-0.07	-0.182
17	-0.000	-0.005	-0.009	-0.019	-0.037	-0.072
19	-0.000	-0.003	-0.005	-0.012	-0.023	-0.039
21	-0.000	-0.002	-0.004	-0.008	-0.015	-0.025

$n\backslash k$	4	6	8	10	12	14
2	-0.018	-0.003	-0.001	0.000	0.000	0.000
4	1.0	0.063	0.020	0.009	0.004	0.003
6	-0.065	1.0	0.079	0.032	0.016	0.009
8	-0.016	-0.083	1.0	0.082	0.036	0.020
10	-0.006	-0.027	-0.088	1.0	0.081	0.038
12	-0.003	-0.012	-0.032	-0.089	1.0	0.08
14	-0.001	-0.006	-0.016	-0.035	-0.089	1.0
16	-0.001	-0.003	-0.009	-0.019	-0.037	-0.087
18	-0.000	-0.002	-0.006	-0.011	-0.021	-0.037
20	-0.000	-0.001	-0.004	-0.007	-0.013	-0.021
22	-0.000	-0.001	-0.002	-0.005	-0.009	-0.014

Added-mass matrices for some particular symmetric edge conditions and homogeneous plates were calculated in the past, see Korobkin and Khabakhpasheva (1999). The present method is general. It is applicable to any complex structures in contact with liquid with minor exceptions, see Khabakhpasheva (2006).

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