Ship Hydroelasticity Analysis with Timoshenko-Beam Approximation Using Legendre and Chebyshev Polynomials

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1 INTRODUCTION

Recently larger and faster ships are being built, so that ships become more flexible and due to forward speed the frequency of encounter in waves approaches the natural frequency of elastic ship motions. Therefore it is necessary to compute wave-induced hydroelastic responses with sufficient accuracy.

Because the Euler-beam model tends to slightly overestimate the natural frequencies of elastic motions and this problem is prominent for higher elastic modes, the present study adopts the Timoshenko-beam approximation which takes into account the deformation due to shearing force. In the structural analysis, it is common to define a special set of natural mode shapes; that is, the dry or wet eigen-modes. Normally FEM may be used for providing mode shapes by taking account of the actual ship geometry, the distribution of flexural rigidity, and the edge boundary conditions. It was noted, however, by Newman (1994) that the structural deflection can be represented instead by a superposition of non-physical mathematical orthogonal functions that are simpler but can predict the physical motions of a body with appropriate boundary conditions satisfied.

Following the idea of Newman, we consider Legendre and Chebyshev polynomials of first kind and second kind as the mode functions for the structural deflection in addition to the dry eigen-modes of the Timoshenko beam. It is shown that a superposition of mathematical polynomial functions can satisfy the required boundary conditions in the process of partial integration for the stiffness matrix and computed results are in good agreement with the results obtained using the dry eigen-modes of a uniform beam, although the rate of convergence is slightly slow with increase in the number of elastic modes when using Chebyshev polynomials.

2 PROBLEM FORMULATION

We consider a ship advancing at constant forward speed U while oscillating with circular frequency of encounter ω_e in a regular wave with amplitude ζ_a, wavenumber k_0, and circular frequency $\omega_0 = \sqrt{gk_0}$. The water depth is assumed deep and the incident angle of wave relative to the positive x-axis is denoted as χ, hence $\omega_e = \omega_0 - k_0 U \cos\chi$. The origin of the coordinate system is placed on the undisturbed free surface and the midst of ship, with positive z-axis taken vertically upward. The ship is considered elastic and thus the ship motion includes not only rigid modes ($j = 1 \sim 6$) but also elastic modes ($j = 7 \sim N$).

The hydrodynamic part of the problem is analyzed with potential-flow assumption, using the velocity potential which is written as

$$\Phi(\boldsymbol{x}, t) = U\Phi_D(\boldsymbol{x}) + \mathrm{Re}\Big[\phi(\boldsymbol{x})\, e^{i\omega_e t}\Big], \quad \phi(\boldsymbol{x}) = \frac{ig\zeta_a}{\omega_0}\Big\{\phi_I(\boldsymbol{x}) + \phi_S(\boldsymbol{x})\Big\} + i\omega_e \sum_{j=1}^{N} X_j \phi_j(\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{x} = (x, y, z)$ and the steady flow is represented by the double-body flow potential Φ_D and additional wavy-flow part is assumed negligible. The spatial part of the unsteady velocity potential $\phi(\boldsymbol{x})$ consists of the components of incident wave ϕ_I, scattered wave ϕ_S and radiation wave ϕ_j ($j = 1 \sim N$) where j denotes the mode of body motion and X_j its complex amplitude.

The unsteady velocity potential ϕ_j ($j = 1 \sim N$ and S) is sought to satisfy the Laplace equation and appropriate boundary conditions on the free surface, the ship-hull surface, and the radiation surface located at a distance from the ship. The free-surface boundary condition to be satisfied on $z = 0$ includes contributions from Φ_D, which is basically the same as that in Sclavounos & Nakos (1990) and hence not written here. The ship-hull boundary condition can be written as

$$\frac{\partial \phi_j}{\partial n} = \begin{cases} \widetilde{n}_j + \dfrac{U}{i\omega_e}\widetilde{m}_j & (j = 1 \sim N) \\ -\dfrac{\partial \phi_I}{\partial n} & (j = S) \end{cases} \quad \text{on } S_H \tag{2}$$

where $$\widetilde{n}_j = \boldsymbol{h}^j \cdot \boldsymbol{n} = h_k^j n_k, \quad \widetilde{m}_j = n_\ell \left(V_k \frac{\partial}{\partial x_k} \right) h_\ell^j - h_\ell^j \left(n_k \frac{\partial}{\partial x_k} \right) V_\ell \tag{3}$$

These are extension of n_j and m_j for the rigid-body motions to the general modes including elastic deflection. The mode vector for the j-th elastic motion is denoted as $\boldsymbol{h}^j = (h_1^j, h_2^j, h_3^j)$. For the case of vertical bending, the components of $\boldsymbol{h}^j$ are expressed with the vertical displacement $w_j(x)$ as follows:

$$h_1^j = -\frac{dw_j(x)}{dx}(z - z_N), \quad h_2^j = 0, \quad h_3^j = w_j(x) \tag{4}$$

where z_N is the vertical position of the neutral axis. V_k in (3) denotes the k-th component of the steady-flow velocity vector $\boldsymbol{V} = \nabla \Phi_D$. The summation signs with respect to k and ℓ are deleted in (3) with Einstein's summation convention.

If the mode function in the j-th mode $w_j(x)$ is specified, a solution of the velocity potential $\phi_j(\boldsymbol{x})$ will be obtained by means of the Rankine Panel Method (RPM) in the frequency domain. Once the velocity potential has been determined, the pressure and resulting hydrodynamic forces in the radiation and diffraction problems can be computed, the results of which will be substituted into the motion equations to determine the complex amplitude X_j of the j-th mode of motion.

3 MOTION EQUATION

In the present study, the deflection of a ship is approximated with the Timoshenko-beam model which includes the distortion by the shearing force in addition to the bending moment. Separating the time dependence $e^{i\omega_e t}$, the spatial part of vertical deflection, denoted as $w(x)$, is governed by the following equation:

$$-\omega_e^2 m w(x) + EI \frac{d^4 w(x)}{dx^4} + \omega_e^2 m \gamma^2 \frac{d^2 w(x)}{dx^2} = f(x) + f^S(x) \tag{5}$$

where $$\gamma^2 \equiv \frac{EI}{k' GA}, \quad f^S(x) \equiv -\gamma^2 \frac{d^2 f(x)}{dx^2} \tag{6}$$

and m is the mass per unit length; EI the flexural rigidity (E is Young's modulus and I the moment of inertia of cross section); GA the shear rigidity (G is the shear modulus and A the cross-section area); k' a constant dependent on the cross-section geometry. Thus γ^2 in (6) is the ratio between the flexural and shear rigidities which has the dimension of length squared. $f(x)$ in (5) denotes the distribution of local pressure force due to an external force on a transverse cross-section of the ship, and the second term $f^S(x)$ defined in (6) is functionally treated as the shearing force.

Since both ends of the ship are free, the boundary conditions for the deflection must be given such that the bending moment and shearing force are equal to zero at both ends of the ship, which are written as

$$\frac{d^2 w(x)}{dx^2} = 0, \quad \frac{d}{dx}\left(EI \frac{d^2 w(x)}{dx^2} \right) = 0 \quad \text{at } x = \pm \frac{L}{2} \tag{7}$$

The spatial part of vertical deflection $w(x)$ may be expanded in an appropriate set of modes as follows:

$$w(x) = \sum_{j=1}^{N} X_j \, w_j(x) \tag{8}$$

where the complex amplitude X_j of each mode is unknown but the same as that in (1), implying that the velocity potential ϕ_j is determined by specifying the mode function $w_j(x)$ in (8).

With the method of weighted residuals, (5) is multiplied by $w_i(x)$ and integrated along the ship's length. The result can be written in the following matrix form:

$$\sum_{j=1}^{N} X_j \left[-\omega_e^2 M_{ij} + \omega_e^2 S_{ij} + D_{ij} \right] = F_i + F_i^S \qquad (i = 1 \sim N) \tag{9}$$

where $$M_{ij} = m \int_{-1}^{1} w_i(q) w_j(q) \, dq, \quad S_{ij} = m\gamma^2 \int_{-1}^{1} w_i(q) \frac{d^2 w_j(q)}{dq^2} \, dq \tag{10}$$

$$D_{ij} = EI \int_{-1}^{1} w_i(q) \frac{d^4 w_j(q)}{dq^4} \, dq = EI \int_{-1}^{1} \frac{d^2 w_i(q)}{dq^2} \frac{d^2 w_j(q)}{dq^2} \, dq \tag{11}$$

and $q = x/(L/2)$ is the normalized coordinate. The right-hand side of (9) includes the diffraction, radiation, and restoring forces due to the pressure force $f(x)$ (which is denoted as F_i) and the shearing force $f^S(x)$ (which is denoted as F_i^S).

The stiffness matrix, D_{ij} given by (11), is transformed using the partial integration. This transformation is correct if each mode function satisfies the free-end boundary conditions (7), like the dry eigen-mode functions to be explained later. If mathematical orthogonal functions like Legendre or Chebyshev polynomials are used in place of the dry eigen-modes of Timoshenko beam, we must enforce the free-end boundary conditions to be satisfied by the sum of mode functions used. This is possible as shown by Newman (1994) and Kashiwagi (1998), in terms of the stiffness matrix represented by the most right-hand side of (11).

4 MODE FUNCTIONS

As the first choice for the mode functions, the dry eigen-modes of a uniform Timoshenko beam are considered, which are homogeneous solutions of (5) and thus written as

$$\frac{1}{\kappa_n^4}\frac{d^4 w_n(x)}{dx^4} + \gamma^2 \frac{d^2 w_n(x)}{dx^2} - w_n(x) = 0, \quad \kappa_n^4 \equiv \frac{m\omega_n^2}{EI} \tag{12}$$

where κ_n denotes the n-th eigen-value associated with the dry-mode frequency ω_n.

Analytical solutions satisfying (12) and the free-end boundary conditions are given as follows:

$$w_{n+5}(x) = \begin{cases} \dfrac{1}{1+(\alpha_n/\beta_n)^2}\left[\dfrac{\cos(\kappa_n\alpha_n q)}{\cos(\kappa_n\alpha_n)} + \dfrac{\cosh(\kappa_n\beta_n q)}{\cosh(\kappa_n\beta_n)}\dfrac{\alpha_n^2}{\beta_n^2}\right], & \text{for } n = 2\ell \\[2ex] \dfrac{1}{1+(\alpha_n/\beta_n)^2}\left[\dfrac{\sin(\kappa_n\alpha_n q)}{\sin(\kappa_n\alpha_n)} + \dfrac{\sinh(\kappa_n\beta_n q)}{\sinh(\kappa_n\beta_n)}\dfrac{\alpha_n^2}{\beta_n^2}\right], & \text{for } n = 2\ell+1 \end{cases} \tag{13}$$

$$\alpha_n = \sqrt{\frac{\sqrt{(\kappa_n\gamma)^4+4} + (\kappa_n\gamma)^2}{2}}, \quad \beta_n = \sqrt{\frac{\sqrt{(\kappa_n\gamma)^4+4} - (\kappa_n\gamma)^2}{2}} \tag{14}$$

where $n = 2, 3, \cdots$; $n = 2\ell$ is an even number and $n = 2\ell + 1$ is an odd number for $\ell = 1, 2, \cdots$; α_n and β_n are given by (14) and thus satisfy $\alpha_n\beta_n = 1$; κ_n denotes the solutions of the eigen-value equation given by

$$\left.\begin{array}{ll} \alpha_n \tan(\kappa_n\alpha_n) + \beta_n \tanh(\kappa_n\beta_n) = 0 & \text{for } n = 2\ell \\ \beta_n \tan(\kappa_n\alpha_n) - \alpha_n \tanh(\kappa_n\beta_n) = 0 & \text{for } n = 2\ell+1 \end{array}\right\} \tag{15}$$

We note that $\kappa_n = 0$ is also a solution of (15) which may be denoted as $\kappa_0 = 0$ and $\kappa_1 = 0$ corresponding to the case of $\ell = 0$ in (15), and these values provide the mode functions of heave and pitch as the rigid-body motions. We also note that the dry eigen-modes of the Timoshenko beam given by (13)–(15) are not orthogonal because of the existence of shearing force. However, for the case of $\gamma^2 = 0$, we can confirm that $\alpha_n = \beta_n = 1$ and thus these solutions become the dry eigen-modes of the Euler beam, which are orthogonal.

In this paper, mathematical orthogonal polynomials are also considered as the mode functions for elastic deflection. One of them is the Legendre polynomials, which can be expressed with Rodrigues' formula in the form

$$w_{n+5}(x) = P_n(q) = \frac{1}{2^n n!}\frac{d^n}{dq^n}(q^2-1)^n, \quad n = 2, 3, \cdots \tag{16}$$

where $q = x/(L/2)$ and hence defined over the interval $-1 \le q \le 1$. These functions are orthogonal and its result can be expressed as

$$\int_{-1}^{1} P_m(q)P_n(q)\,dq = \frac{2}{2n+1}\,\delta_{mn} \tag{17}$$

As another orthogonal polynomials, the Chebyshev polynomials are considered, which include the first kind denoted as $T_n(q)$, and the second kind denoted as $U_n(q)$. These are expressed as follows:

$$w_{n+5}(x) = T_n(\cos\theta) = \cos n\theta, \quad w_{n+5}(x) = U_n(\cos\theta) = \frac{\sin(n+1)\theta}{(n+1)\sin\theta} \tag{18}$$

where $q = \cos\theta$ and $n = 2, 3, \cdots$. It should be noted that the second kind $U_n(q)$ is modified from the original definition by dividing with $n + 1$ so that $U_n(1) = 1$. The Chebyshev polynomials are also orthogonal in terms of weight function and its result can be expressed as

$$\int_{-1}^{1} T_m(q)T_n(q)\frac{dq}{\sqrt{1-q^2}} = \frac{\pi}{2}\,\delta_{mn}, \quad \int_{-1}^{1} U_m(q)U_n(q)\sqrt{1-q^2}\,dq = \frac{\pi}{2}\frac{1}{(n+1)^2}\delta_{mn} \tag{19}$$

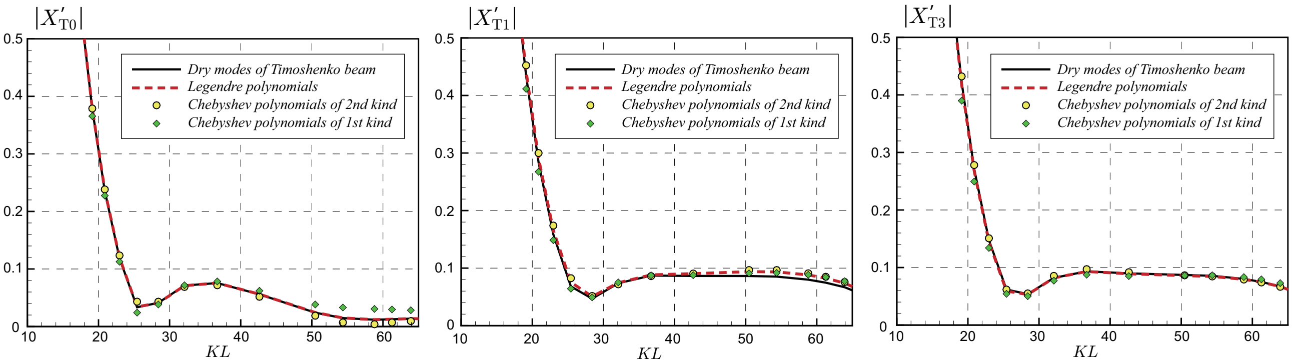
However, these orthogonal relations of Chebyshev polynomials cannot be applied to computation of the mass matrix defined by (10) and also the stiffness matrix defined by (11) owing to weight functions shown in (19). Therefore it is worth noting that there is coupling in the mass matrix for all even or odd modes, including the coupling of rigid modes with symmetric or antisymmetric polynomials for elastic modes.

5 RESULTS AND DISCUSSION

5.1 Comparison with experiments at zero speed

As the first validation example, computed results are compared with the experiment conducted by Malenica *et al.* (2003), measuring the vertical deflection of an elastic barge model which is composed of 12 small floaters. They are connected by two long plates on the top of the barge that have a low flexural rigidity.

Although the results are not shown here owing to paucity of space, obtained results are in good agreement with experimental data, and noticeable difference cannot be seen between the results using the Legendre polynomials and the dry eigen-modes. Through comparison of the results between Euler beam ($\gamma^2 = 0$) and Timoshenko beam, the effect of shearing force is confirmed to be negligible in the present case.



(a) $|X'_3+X'_5|$ (b) $|X'_3+X'_5+X'_7|$ (c) $|X'_3+X'_5+X'_7+X'_8+X'_9|$

Fig. 1 Nondimensional amplitude of the vertical deflection at the bow ($q = 1$), summation of rigid and elastic modes of motions, computed at $Fn = 0.2$ for the modified Wigley model and $KL = \omega_e^2 L/g$.

5.2 Convergence study at forward speed

As the next validation, convergence in computed results with increasing the number of modes is studied for the forward-speed case ($Fn = 0.2$) using the RPM and a modified Wigley model adopted in Kashiwagi *et al.* (2015). To see the convergence in the total deflection of the model particularly at the bow ($q = 1$), we define the following value, nondimensionalized with incident-wave amplitude ζ_a:

$$\left|X'_{Tj}\right| = \left|X'_3 + X'_5 + \sum_{i=7}^{6+j} X'_i\right| \tag{20}$$

The value $\left|X'_{T0}\right|$ means only the rigid motion (heave and pitch), and subscript j increasing, more elastic modes are added to present the total vertical deflection at the bow ($q = 1$).

For rigid motions shown in Fig. 1(a), the results using the Legendre polynomials and the dry eigen-modes of Timoshenko beam are in virtually perfect agreement, since the coupling coefficients in the matrix between rigid and elastic modes are basically all zero in both methods. On the other hand, when using the Chebyshev polynomials of first kind and second kind, we can see a clear difference from the other results in a range of higher frequency because of relatively large values in the coupling matrix coefficients. This is because the Chebyshev polynomials are simply used as the mode functions in computing (10)–(11) without the weight functions unlike in (19). Fig. 1(b) includes the first elastic bending mode X_7, in which a noticeable difference exists, implying that the convergence is not achieved yet. In fact, in Fig. 1(c) adding up to three elastic modes, no visible difference exists in the results between Legendre polynomials and dry eigen-modes. However, we can see still slight discrepancy when using the Chebyshev polynomials, which may be reduced to practically zero with increasing more the number of modes.

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