

# Prediction of Bi-directional Water Waves Using Approximate Impulse Response Function

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## 1 INTRODUCTION

Accurate short-term predictions of water waves are demanded for forecasting a ship motion (Naaijen *et al.*, 2018), harvesting a wave energy absorber (Al Shami *et al.*, 2019), and so on. Although a prediction for head waves is important for moving ships, a prediction for multi-directional waves is of great importance for offshore structures because these structures are exposed to multi-directional waves. In this paper, we propose a method to predict free surface displacement on bi-directional waves, consisting of progressive and regressive waves. The prediction model is based on a Linear Time-Invariant (LTI) system. Since a conventional impulse response function of water waves is obtained under the assumption of deep water throughout frequencies (Davis and Zarnick, 1966 and Falnes, 1995), prediction errors are excessive due to overestimation of non-causality. Therefore, the dispersion of finite-depth water is considered to reduce the non-causal effect, and an approximate solution of the impulse response function of finite-depth water waves is proposed. A tank experiment is also carried out to validate proposed methods, and a prediction of irregular waves with the JONSWAP spectrum is compared with that of the experimental result.

## 2 APPROXIMATE IMPULSE RESPONSE FUNCTION

A two-dimensional problem with horizontal and vertical planes, as shown in Fig.1 (a), is considered. All parameters are normalized by a constant water depth  $d$ , the gravitational acceleration  $g$  and an amplitude of waves, and non-dimensional parameters are shown in the paper. The theory is developed based on the LTI system. When only progressive waves (propagating from left-hand side to right-hand side) are considered, the free surface displacement at the point  $P$  is predicted using a convolution integral of a time history of the displacement at the point  $A$  and an impulse response function  $h(t)$  as

$$\xi_P = \int_{-\infty}^{\infty} h(\tau)\xi_A(t - \tau)d\tau \quad (1)$$

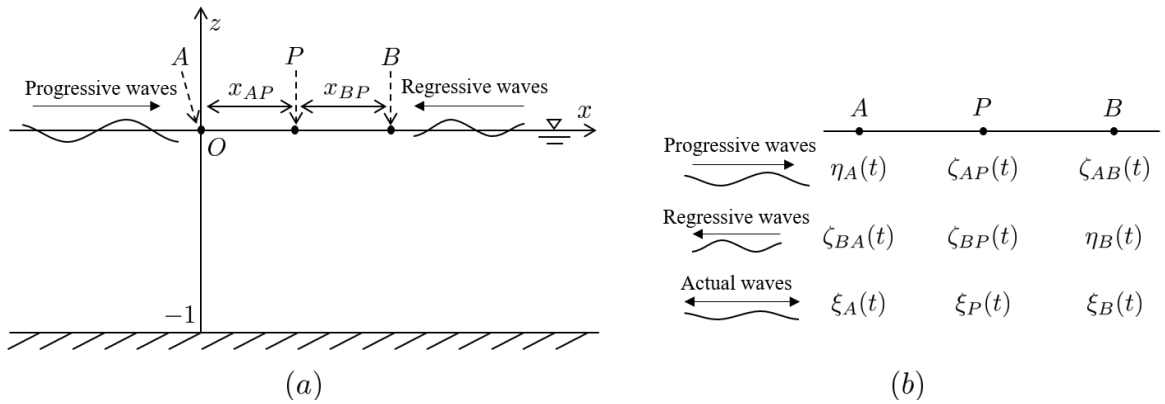
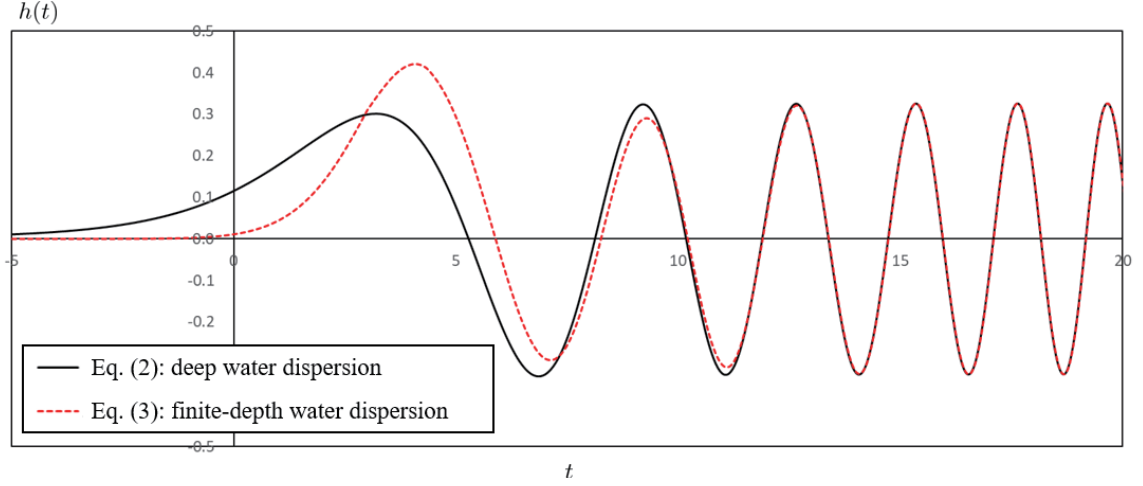


Fig. 1 (a) Schematic view of bi-directional water wave problem. Free surface displacement at point  $P$  is predicted from time histories of waves at points  $A$  and  $B$ . (b) Decomposition of bi-directional waves to progressive and regressive waves.



**Fig. 2 Impulse response functions of deep water and finite-depth water waves. Results of  $x_{AP} = 3$  are shown.**

If the impulse response function is causal, the integral range can be replaced by  $[0, \infty]$ . The impulse response function of linear water wave are given by the Fourier transform of a frequency response function  $H(\omega) = \exp(ikx_{AP})$ , where  $\omega$  is a natural frequency and  $k$  is a wave number. According to Davis and Zarnick (1966), the impulse response function with a dispersion relation of deep water waves  $\omega|\omega| = k$  is analytically obtained as

$$h_D(t) = \sqrt{\frac{1}{2\pi x_{AP}}} \left[ \cos \frac{t^2}{4x_{AP}} \left\{ \frac{1}{2} + \mathcal{C} \left( \sqrt{\frac{1}{2\pi x_{AP}}} t \right) \right\} + \sin \frac{t^2}{4x_{AP}} \left\{ \frac{1}{2} + \mathcal{S} \left( \sqrt{\frac{1}{2\pi x_{AP}}} t \right) \right\} \right] \quad (2)$$

where  $\mathcal{C}(\cdot)$  and  $\mathcal{S}(\cdot)$  are the cosine and sine forms of the Fresnel integral, respectively. This function is available for predictions of deep water waves. Since this function is non-zero at  $t = 0$ , this function is non-causal. Therefore, using current and past data of the wave profile at the point  $A$  is imperfect to predict waves at the point  $P$ . This non-causal effect decays as the distance with  $x_{AP}^{-1/2}$  and this vanishes as  $x_{AP} \rightarrow \infty$ , so that long distance is needed for accurate predictions.

The reason of non-causality of water waves was studied by Falnes (1995), and it was concluded that dispersive waves cannot be causal because of the mathematical formulation of the problem. This must be true that the impulse response function  $h(0) = (1/2\pi) \int_{-\infty}^{\infty} c_g(k) \cos(kx_{AP}) dk$  is never zero because the group velocity  $c_g = d\omega/dk$  modulates the integrand if dispersive. Non-causality is unavoidable, however, the use of the dispersion of deep water waves excessively overestimates the non-causal effect. The group velocity of deep water waves approaches infinity ( $c_g \rightarrow \infty$ ) as the wave number is zero ( $k \rightarrow 0$ ), and a contribution to  $h_D(0)$  is dominant around  $k \approx 0$ . This results the overestimation of the non-causal effect. Since the maximum group velocity is bound by the shallow water limitation  $c_g = 1$ , the actual non-causal effect should be less than that of the deep water assumption. Therefore, the use of the dispersion of finite-depth water waves  $\omega|\omega| = k \tanh k$ , instead of the deep water dispersion, is proposed to reduce the non-causal effect.

Since it is difficult to obtain analytical solution of the impulse response function of finite-depth water waves, we propose an approximate solution consisting of three forms with respect to time. Final result is given as

$$h_F(t) = \begin{cases} \left( \frac{2}{x_{AP}} \right)^{\frac{1}{3}} \text{Ai}[\alpha] & t \leq t_0 \\ a(t - t_0)^3 + b(t - t_0) + h_F(t_0) & t_0 < t < t_1 \\ \text{Re} \left[ \frac{1}{\pi} c_g(k_0) \sqrt{\frac{2\pi}{t|\omega''(k_0)|}} e^{i[\omega(k_0)t - k_0 x_{AP} - \frac{\pi}{4}]} \right] & t_1 \leq t \end{cases} \quad (3)$$

where

$$\begin{aligned}\alpha &= (x_{AP} - t) \left( \frac{2}{x_{AP}} \right)^{\frac{1}{3}} \\ a &= -\frac{1}{2(t_2 - t_0)(t_1 - t_0)} \left[ \frac{h(t_1)}{t_2 - t_1} + \frac{h(t_1) - h(t_0)}{t_1 - t_0} \right] \\ b &= \frac{h(t_1) - h(t_0)}{t_1 - t_0} + \frac{t_1 - t_0}{2(t_2 - t_0)} \left[ \frac{h(t_1)}{t_2 - t_1} + \frac{h(t_1) - h(t_0)}{t_1 - t_0} \right] \\ t_0 &= x_{AP}, \quad t_1 = t|_{\phi=\pi/4}, \quad t_2 = t|_{\phi=3\pi/4}\end{aligned}$$

Here,  $\text{Ai}[\cdot]$  is the Airy function and  $\phi \equiv \omega t - kx_{AP}$  is a phase function. To obtain these three forms, the Maclaurin series expansion, the spline interpolation and the stationary phase method are employed. Details of calculation processes are presented at the workshop. Comparison of the impulse response functions between the deep water dispersion and the finite-depth water dispersion is shown in Fig.2.

### 3 DECOMPOSITION OF BI-DIRECTIONAL WATER WAVES

If only progressive waves are traveling on the free surface, measuring the wave time history at the point  $A$  is sufficient to predict waves at the point  $P$ . Nevertheless, this situation is rare even in the tank experiment, i.e. regressive waves are coming from the opposite side due to the reflection at the wall. To achieve the prediction of bi-directional water waves, an additional point  $B$  is put on the opposite side of the point  $A$  as in Fig.1 (a). Using these two points  $A$  and  $B$ , waves at the point  $P$  is predicted.

Fig.1 (b) shows the decomposition of bi-directional waves to progressive and regressive waves, and the definition of each wave component.  $\eta_A(t)$  represents progressive waves at the point  $A$ , while  $\eta_B(t)$  is regressive waves at the point  $B$ .  $\zeta_{mn}(t)$  is predicted waves propagating from  $m$  to  $n$  direction, where  $n$  ( $= A, B, P$ ) is the predicted point and  $m$  ( $= A, B$ ) is the input point.  $\xi_n(t)$  is actual waves which is obtained by the superposition of progressive and regressive waves at the point  $n$ . The prediction time range in eq.(1) is truncated to a causal and finite range  $[0, T]$ . Then, predicted wave component  $\zeta_{mn}(t)$  is discretized as

$$\zeta_{mn}(t) \approx \int_0^T h_{mn}(\tau) \eta_m(t - \tau) d\tau \rightarrow \zeta_{mn}^{(0)} \equiv a_{mn}^{(0)} \eta_m^{(0)} + b_{mn} \quad (4)$$

where

$$\begin{aligned}a_{mn}^{(0)} &= h_{mn}^{(0)} \frac{\Delta\tau}{3} \\ b_{mn} &= \frac{\Delta\tau}{3} \left[ 4 \sum_{\ell=1}^{L/2} h_{mn}^{(2\ell-1)} \eta_m^{(2\ell-1)} + 2 \sum_{\ell=1}^{L/2-1} h_{mn}^{(2\ell)} \eta_m^{(2\ell)} + h_{mn}^{(L)} \eta_m^{(L)} \right]\end{aligned}$$

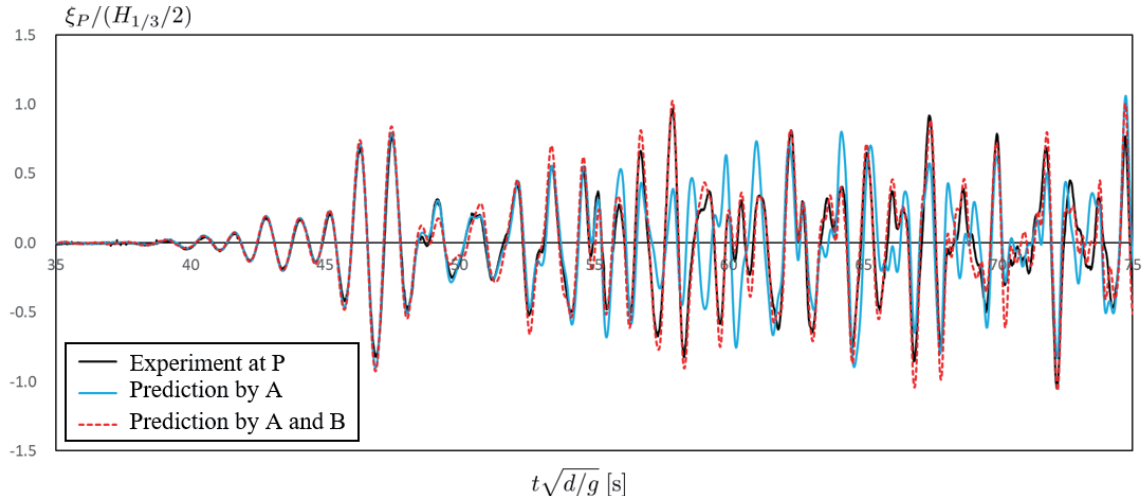
Here, superscript (0) means values at the current time. In numerical integration, Simpson's rule is used where  $L + 1$  is the number of discretized points. The constant  $b_{mn}$  is calculated by  $\eta_m^{(\ell)}$  at the past times, and thus this is known value. Since actual waves at the points  $A$  and  $B$  at the current time  $\xi_n^{(0)}$  are expressed by the superposition of  $\eta_m^{(0)}$  and  $\zeta_{mn}^{(0)}$ , unknown values  $\eta_m^{(0)}$  are given as

$$\begin{bmatrix} \eta_A^{(0)} \\ \eta_B^{(0)} \end{bmatrix} = \frac{1}{1 - a_{AB}^{(0)} a_{BA}^{(0)}} \begin{bmatrix} (\xi_A^{(0)} - b_{BA}) - a_{BA}^{(0)} (\xi_B^{(0)} - b_{AB}) \\ -a_{AB}^{(0)} (\xi_A^{(0)} - b_{BA}) + (\xi_B^{(0)} - b_{AB}) \end{bmatrix} \quad (5)$$

Once  $\eta_m^{(0)}$  is obtained, waves at the point  $P$  are predicted as

$$\xi_P^{(0)} = \zeta_{AP}^{(0)} + \zeta_{BP}^{(0)} = a_{AP}^{(0)} \eta_A^{(0)} + b_{AP} + a_{BP}^{(0)} \eta_B^{(0)} + b_{BP} \quad (6)$$

Note that this is a time evolution problem and thus an initial condition for  $\eta_m^{(\ell)}$  is necessary.



**Fig. 3** Time history of irregular waves based on JONSWAP spectrum at point  $P$ . Comparison among experimental result and predicted results.

## 4 EXPERIMENTAL AND SIMULATION RESULTS

To validate the proposed theories, a tank experiment was carried out. The experiment was conducted in a two-dimensional tank at Osaka University, Japan. The length of the tank is 14 [m] and the width is 0.3 [m]. The tank is filled with pure water until the water level  $d = 0.45$  [m]. The distances among points  $A$ ,  $P$  and  $B$  are  $x_{AP}d = x_{PB}d = 1.5$  [m]. The sampling time of the wave gauges is 0.01 [s]. Irregular waves based on the JONSWAP spectrum with  $\gamma = 3.3$  are generated by the wave maker. The significant height is  $H_{1/3} = 0.03$  [m]. Note that this height is not the height of the free surface but the height of the motion of the wave maker. The significant period is set as  $T_{1/3}\sqrt{d/g} = 1.2$  [s]. Irregular waves are made up from various types of waves e.g., shallow, finite-depth and deep water waves. For the initial condition of the prediction,  $\eta_m^{(\ell)} = 0$  is used. The predicted waves at the point  $P$  with causal range  $[0, T\sqrt{d/g}] = [0, 10]$  (second) and the time step  $\Delta t\sqrt{d/g} = 0.01$  [s] are shown in Fig.3. Predicted result using only the data of the point  $A$  shows large discrepancy with the experimental result after  $t\sqrt{d/g} = 55$  [s] due to reflected waves. On the other hand, the use of two points  $A$  and  $B$  agrees with the experimental time history even regressive waves are coming. Although a phase is in good agreement, a peak value is sometimes different from that of the experiment. Especially, accuracy of the prediction by the points  $A$  and  $B$  is worse than that of the point  $A$  around  $t\sqrt{d/g} = 49$  [s]. Since actual waves are decomposed into progressive waves and regressive waves based on the non-causal impulse response function, the non-causal effect might be accumulated rather than the prediction by single point. Further discussions are shown in the workshop.

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