# Effect of wave nonlinearity on the statistics of wave groups and extreme ship motions

Xianliang Gong[1], Kevin M. Silva[1,2], Kevin J. Maki[1], and Yulin Pan[1]

[1]Department of Naval Architecture and Marine Engineering
University of Michigan, Ann Arbor, MI 48109 USA

[2]Surface Ship Hydromechanics Simulations and Analysis Branch
Naval Surface Warfare Center Carderock Division, West Bethesda, MD 20817 USA

## INTRODUCTION

The statistics of ocean wave groups is a problem of long interest to oceanographers and marine engineers (the latter due to its relevance to, say, ship motion statistics). The critical wave group (CWG) method [1, 3] has been developed as an advanced approach to evaluate the extreme ship motion probability induced by an ensemble of wave groups. The centralized idea in the CWG method is that the wave-group probability (or probability that a wave group exceeds a threshold) is established assuming the Markov chain property of adjacent wave heights. The CWG method constructs wave groups based on statistics that can be derived from either linear or nonlinear waves, and it has been mostly studied using statistics derived from linear wave fields. The effect of wave nonlinearity on both the wave group and ship motion statistics has not been studied in detail. In this work, we specifically study the effect of wave nonlinearity on wave group statistics in nonlinear wave fields generated by the high-order spectral (HOS) method [2, 4]. Extreme ship motion is then evaluated through a nonlinear roll equation using the CWG method. We show that, for a narrow-band wave field with large steepness, the extreme motion (and the wave group) probability can be significantly influenced by the wave nonlinearity.

## METHODOLOGY

We sample both linear and nonlinear wave fields described by the same JONSWAP spectrum to determine the conditional probabilities of wave height and period of successive waves, as well as the associated extreme ship motion statistics.

**HOS** While the linear wave field can be generated by assigning random phases to each wave mode, the nonlinear wave field is generated through the HOS computations starting from linear wave fields. Specifically, HOS solves the nonlinear wave equations:

$$\frac{\partial \eta(x,t)}{\partial t} + \frac{\partial \psi(x,t)}{\partial x} \cdot \frac{\partial \eta(x,t)}{\partial x} - [\frac{\partial \eta(x,t)}{\partial x} \cdot \frac{\partial \eta(x,t)}{\partial x}]\phi_z(x,t) = 0 \tag{1}$$

$$\frac{\partial \psi(x,t)}{\partial t} + \frac{1}{2}\frac{\partial \psi(x,t)}{\partial x} \cdot \frac{\partial \psi(x,t)}{\partial x} + \eta(x,t) - \frac{1}{2}[1 + \frac{\partial \eta(x,t)}{\partial x} \cdot \frac{\partial \eta(x,t)}{\partial x}]\phi_z^2(x,t) = 0 \tag{2}$$

where $x$, $z$ and $t$ are the horizontal and vertical coordinates, and time, $\eta(x,t)$ is the surface elevation, $\psi(x,t)$ the surface potential, $\phi_z(x,t)$ the surface vertical velocity with $\phi(x,z,t)$ the velocity potential of the flow field. As studied in [5], the nonlinear wave effect is manifested (in terms of its influence to wave statistics) in 40-60$T_p$, with $T_p$ the peak period of the spectrum. We therefore take an ensemble of time series (from different spatial locations and ensemble of runs) in 40-60$T_p$, based on which we evaluate the wave group statistics in nonlinear wave fields. We assume that the spectral evolution in 40-60$T_p$ only has secondary effect in modifying the group statistics.

**Critical Wave Group Method** To compute the probability of extreme ship motion, the CWG method expresses the probability of a response $\phi$ exceeding a threshold $\phi_{\text{crit}}$ as the probability of all wave groups and initial condition (i.e., the state of the ship at the moment of encounter,) combinations that lead to an exceedance. A critical wave group, defined as a group of waves that leads the ship to an infinitesimal exceedance of a response threshold, is mathematically described by a vector $\mathbf{h}_{cr}^{(k)}$ that contains the heights of $j$ waves and their corresponding periods $\mathbf{T}_j$.

The CWG method determines the probability of threshold exceedance as

$$p\left[\phi > \phi_{\text{crit}}\right] \;=\; \sum_k \sum_m \left(1 \;-\; \prod_j \left(1 \;-\; p\left[wg_{m,j}^{(k)}\right]\right)\right) \;\times\; p\left[ic_k\right], \tag{3}$$

where $p\left[wg_{m,j}^{(k)}\right]$ is the probability of observing a wave group that is larger than the critical wave group, with indices $m$, $j$ and $k$ denoting the wave period, number of waves and initial conditions. $p\left[ic_k\right]$ is the probability of the initial condition $ic_k$. The key computation involved in (3) is the term $p\left[wg_{m,j}^{(k)}\right]$, which can be formulated by:

$$p\left[wg_{m,j}^{(k)}\right] \;=\; p\left[\mathbf{H}_j > \mathbf{h}_{cr,j}^{(k)}, \mathbf{T}_j \in T_{cr,m}\right]. \tag{4}$$

This probability is determined by evaluating the joint probability of the wave heights $\mathbf{H}_j$ exceeding the heights of the critical wave group $\mathbf{h}_{cr}^{(k)}$, and the wave period $\mathbf{T}_j$ being within the range $T_{cr,m}$. In this work, the joint probability of successive wave heights and periods is determined from the Markov chain probability (of adjacent two waves), extracted from wave fields generated by linear theory and nonlinear HOS computations. This Markov chain approach allows each individual wave group to be uniquely described by $H_c$, $T_c$ (which are the height and period of the largest wave), and $j$.

The probability of initial conditions $p\left[ic_k\right]$ is obtained by simulation of ship motion in random irregular waves and developing a probabilistic distribution of the quantities of interest. The initial condition distribution can then be discretized into $k$ initial states for the identification of critical wave groups and computation of the exceedance probability.

**Roll Model** Since our focus is on the nonlinear wave effect, we use a simplified model to compute the ship roll motion in a given wave group, namely a phenomenological nonlinear roll equation [1]:

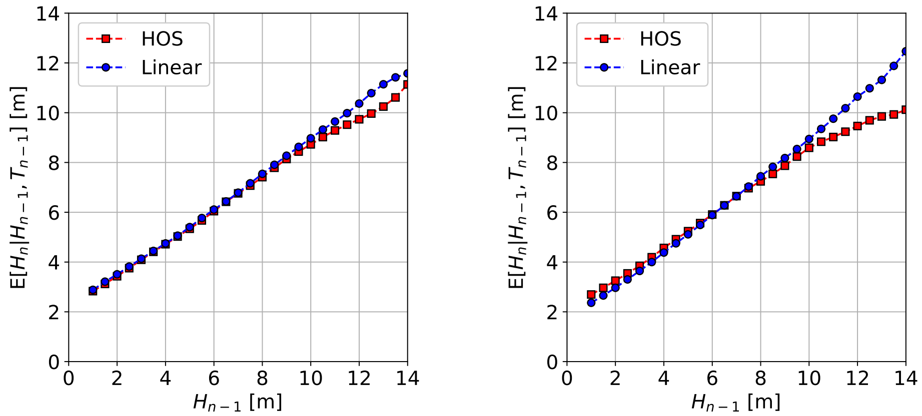
$$\ddot{\phi} + \alpha_1\dot{\phi} + \alpha_2\dot{\phi}|\dot{\phi}| + \beta_1\phi + \beta_2\phi^3 = \epsilon\eta(t), \tag{5}$$

where $\eta(t)$ is the time series of wave elevation, and the solution $\phi(t)$ represents the resulting roll motion of the ship. We use empirical parameters $\alpha_1 = 0.095$, $\alpha_2 = 0.052$, $\beta_1 = 0.1175$, $\beta_2 = -0.09$, $\epsilon = 0.009$. The wave elevation $\eta(t)$ is determined from the critical wave group parameters using a Fourier basis.

## RESULTS

We apply the CWG method on the system described in Eqn. 5 that is operating in a JONSWAP spectrum with significant wave height $H_s = 7.5$ m, peak wave period $T_p = 12$ s and peak enhancement factor $\gamma = 9$. We sample from linear wave fields and HOS simulations of order three to compute wave group statistics.

We first test the probabilistic relation between successive waves for both linear and nonlinear waves. Treating wave successions as a Markov chain, the most likely following height and period can be found by utilizing their conditional probability given current wave's height. Fig. 1 shows the expected (most likely) following wave height $H_n$, given the present wave height $H_{n-1}$, for waves of period 12 and 13 s. For both periods, wave heights of less than 10 m show that the HOS and linear wave theories predict similar behavior, whereas for waves greater than 10 m, the most likely following wave height is over-predicted by the linear theory.

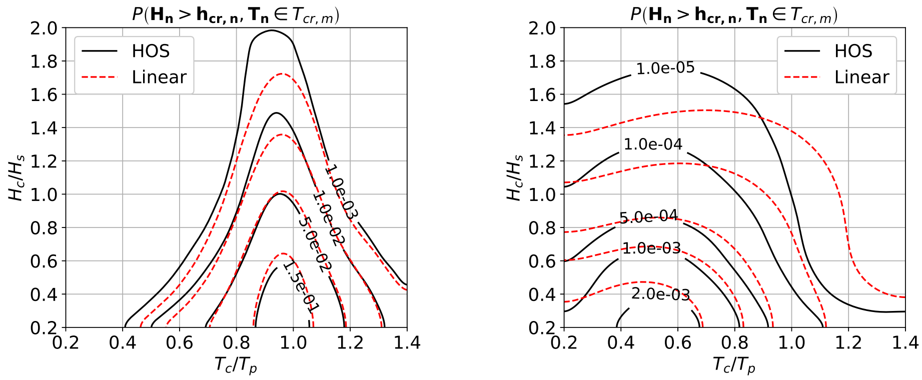



(a) $T_{n-1} = 12$ s

(b) $T_{n-1} = 13$ s

Figure 1: Markov chain predictions of the expected following wave height given the present wave height ($H_{n-1}$) and period ($T_{n-1}$).

Ultimately, the probability of exceedance depends on the probability of a wave group being larger than the critical wave group. Fig. 2 shows a comparison for HOS and linear waves of the probability of wave group exceedance described in Eqn. 4 for a single wave ($j = 1$) and two successive waves described by the Markov chain prediction ($j = 2$).



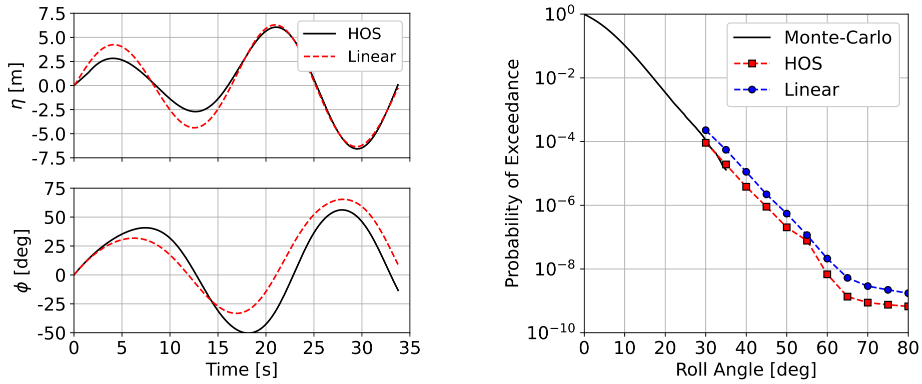

(a) $j = 1$

(b) $j = 2$

Figure 2: Comparison of the probability of wave group exceedance for linear and nonlinear wave fields.

For $j = 1$, the contours of Fig. 2(a) represent the probability of any single wave with a given period $T_c$ exceeding the height $H_c$. The probability contours demonstrate that encountering larger single waves is more probable using the HOS statistics. For $j = 2$, the contours in Fig. 2(b) represent the probability that any two successive waves, where the period of the largest wave is $T_c$, exceeds a two-wave group with the largest wave $H_c$ and the successive wave determined by the Markov chain prediction. The HOS contours trend towards higher probability for lower period wave groups while the linear statistics trend towards higher probabilities for larger wave periods. This dependence on period is consistent with Fig. 1, where the Markov chain predictions of following wave height show larger differences for $T_c = 13$ than $T_c = 12$.

Fig. 3 shows a comparison of the nonlinear and linear cases for an example wave group, as well as the probability of exceedance at various roll thresholds. Fig. 3(a) demonstrates that for a $j = 2$ wave group with the same largest wave, and initial conditions, the resulting roll response is different. This difference is directly linked to the phenomena shown in Fig. 1 and is purely from the construction of the wave groups. Fig. 3(a) highlights that the different prediction of the most likely following wave from the nonlinear and linear wave statistics results in a different maximum roll response.




(a) $H_c = 13$ m, $T_c = 17$ s, $j = 2$ (b) Probability of exceedance

Figure 3: Comparison of a sample roll time history for a similar $j = 2$ wave group and of probability of exceedance utilizing statistics from linear and nonlinear wave fields.

The over-prediction of the following wave height in Fig. 1 also leads to an over-prediction of the roll response and thus, an over-prediction in the probability of exceedance shown in Fig. 3(b). Another contribution to over-prediction of the probability of exceedance for the linear wave field is that the probability of wave group exceedance for $j = 2$ in Fig. 2 is higher for larger periods in the linear case. Additionally, Eqn. 5 has an estimated natural period of 18.3 s. Thus, the over-prediction of probability of exceedance for the given case study can be attributed to a combination of the over-prediction of the following wave in the Markov chain predictions that becomes more evident as the wave period increases, the trending towards larger periods in the probability of wave group exceedance for linear wave fields, and the natural period of roll residing in a region where the probability of wave group exceedance is larger with the linear wave statistics. This study indicates that the differences between the resulting probability of exceedance for nonlinear and linear wave fields may not always be consistent with being either an over-prediction or an under-prediction. The natural period of the response will also play a role in the trends.

The preliminary results show that the conditional probability of successive waves as determined from linear or nonlinear waves is different. Also, the difference is notable for the extreme motion probability. If the abstract is accepted, the authors will study and present results for the extreme roll motion of a ship section using Computational Fluid Dynamics (CFD). The ship section will be from the midship section of the ONR Tumblehome hull form.

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