Sloshing and scattering in shallow water

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Abstract

The number of explicit solutions to the linear shallow-water equation with a variable depth is small. Such solutions involve reducing the governing equation to one involving special functions whose properties are well established. Here we introduce what is believed to be a new solution in terms of Bessel functions and discuss an existing solution in terms of elementary functions. We also develop a connection between problems of sloshing in containers and scattering by submerged obstacles providing general results for arbitrary depth variation which we then apply in a limited way to specific cases.

General Theory

Using the linear shallow-water approximation, the free surface displacement $\eta(x)$ satisfies

$$\frac{d}{dx}\left(h(x)\frac{d\eta(x)}{dx}\right)+\frac{\omega^2}{g}\eta(x)=0 \tag{1}$$

where ω is the radian frequency and we assume the bottom topography $h(x)$, is measured downwards. One of the few examples where the choice of $h(x)$ results in an explicit solution of (1) for $0\le x\le a$ is given by $h(x)=(hx/a)$ where h is constant. Then the substitution $t^2=x/a$ reduces (1) to Bessel's equation with solution $J_0(2ka(x/a)^{1/2})$. The second solution is rejected as being unbounded at $x=0$. Reflection in $x=a$ then provides the solution for the natural modes of oscillation in a wedge-shaped basin of depth h and water-line width $2a$. The natural frequencies of the oscillations divide into two types, being the roots of $J_0(2ka)=0$ and $J_1(2ka)=0$ corresponding to the modes being odd and even about $x=a$ respectively. Here $k^2=\omega^2/gh$. A comprehensive review of sloshing problems is given in Faltinsen & Timokha (2009).

This solution can be used to include a wider class of problems, and also extended to cover a more general $h(x)$. For example if we assume that $h(x)=(hx/a)$ for $0\le x\le a$ and $h(x)=h$, $a\le x\le b$, $b\ge a$, then reflection in $x=b$ produces a trough of depth h and width $2b$ at the surface sloping down uniformly to width $2c=2(b-a)$ at the bottom. The natural frequencies of the oscillations in the trough again divide into two types, corresponding to the odd and even modes about $x=b$ given by $C\sin k(b-x)$ and $C\cos k(b-x)$ respectively. Matching these modes with $\eta(x)$ and $\eta'(x)$ across $x=a$ gives

$$\tan kc=J_0(2ka)/J_1(2ka),\quad =-J_1(2ka)/J_0(2ka) \tag{2}$$

for the odd and even resonant conditions, reducing to the result for the wedge-shaped basin when $c=0$.

For a more general topography we choose $h(x)=h(x/a)^r$, $0<r<1$ for $0\le x\le a$ which, after reflection about $x=b$, describes a trough with curved ends intersecting the free surface at $x=0,\ 2b$ vertically. Let $t^s=(x/a)$ so that (1) becomes

$$\frac{d}{dt}\left(t^{s(r-1)+1}\frac{d\eta(t)}{dt}\right)\frac{1}{t^{s-1}}+\kappa^2\eta(t)=0$$

where $\kappa=kas$. If now we assume $s(r-1)+1=0$ so that $t=(x/a)^{1-r}$, then we have

$$\frac{d^2\eta(t)}{dt^2}+\kappa^2t^m\eta(t)=0,\quad m=r(1-r)^{-1}$$

with solution (Gradshteyn & Ryzhik (1965) $\mathrm{p.971, 8.491(7)}$) in terms of x,

$$\eta(x) = (x/a)^{(1-r)/2} J_\nu(2ka(x/a)^{1-r/2}/(2-r)), \quad \nu = (1-r)/(2-r) \tag{3}$$

for $0 < r \leq 1$ where we have chosen the solution which is bounded as $x \to 0$ since $\eta(x) \sim x^{1-r}$, $x \to 0$, the other solution ruled out as being singular at $x = 0$. We have

$$\eta(a) = J_\nu(2ka/(2-r)), \quad 2a\eta'(a) = (1-r)J_\nu(2ka/(2-r)) + 2kaJ'_\nu(2ka/(2-r))$$

so the resonant frequencies in the trough are given by

$$\frac{(1-r)}{2ka} + \frac{J'_\nu(2ka/(2-r))}{J_\nu(2ka/(2-r))} = -\cot kc, \quad \text{or } \tan kc \tag{4}$$

For $r = 1$ equation (3) reduces to $\eta(x) = J_0(2ka(x/a)^{1/2})$ and the resonant conditions reduce to $J'_0(2ka)/J_0(2ka) = -\cot kc$, or $\tan kc$, in agreement with (2). Finally when $c = b - a = 0$ we recover the conditions $J_0(2ka) = 0$ and $J'_0(2ka) = 0$ in agreement with the known wedge-shaped solution.

We can generalise this bottom shape by assuming $h(x)$ is defined by

$$h(x) = h^{(r)}(x) = h_1(1 + \beta x/a)^r, \quad 0 \leq x \leq a, \quad (1+\beta)^r = h_2/h_1 \tag{5}$$

for $0 < r \leq 1$ and for $r = 2$, so that $h^{(r)}(0) = h_1$, $h^{(r)}(a) = h_2$ where $0 \leq h_1 \leq h_2 \leq h$. For $r = 1$ the bottom slope is constant from a depth h_1 at $x = 0$ to h_2 at $x = a$. For $0 < r < 1$ the bottom is a curve with slope at $x = 0$ greater and at $x = a$ less than for $r = 1$ with the converse being true if $r = 2$. We find, after substituting $t = (1 + \beta x/a)^{1-r}$ that in terms of x,

$$\eta(x) = ((1 + \beta x/a)^{(1-r)/2} Z_\nu(2k_1 a(1 + \beta x/a)^{1-r/2}/\beta(2-r)), 0 < r \leq 1 \tag{6}$$

where $k_1^2 h_1 = k^2 h$ and $\nu = (1-r)/(2-r)$, and $Z_\nu(z)$ stands for the Bessel functions $J_\nu(z)$, $Y_\nu(z)$ or any linear combination of them.

If we now assume that for $x \geq a$, $h^{(r)}(x) = h$, a constant, and $h^{(r)}(x)$ is symmetric about $x = 0$, then by reflection the bottom boundary becomes a submerged mount in water of depth h in the shape of a rectangle with vertical sides extending down from a depth h_2 to h topped by the particular $h(x)$ in $0 \leq x \leq a$ given by (5).

Thus we have a scattering problem for all x so that

$$\eta(x) = e^{-ikx} + Re^{-ikx} \quad x \geq a, \quad \eta(x) = Te^{-ik(x+a)} \quad x \leq -a. \tag{7}$$

Because of symmetry we can write $\eta(x) = \eta_0(x) + \eta_1(x)$ where $\eta_0(x)$ and $\eta_1(x)$ are odd and even respectively about $x = 0$ so that $\eta_0(0) = 0$, $\eta'_1(0) = 0$. Then it follows that $R = (R_0 + R_1)/2$ and $T = (R_0 - R_1)/2$ where the R_i, $(i = 0, 1)$ satisfy the first equation in (7). Matching $\eta_i(x)$ and flux across $x = a$ results in

$$R_i = e^{-2i\theta_i} \quad \text{where} \quad \tan\theta_i = \delta_i = \left(\frac{h_2}{kh}\right)\frac{\eta'_i(a)}{\eta_i(a)} \quad (i = 0, 1) \quad \text{so that} \tag{8}$$

$$|R|^2 = \frac{(1 + \delta_0\delta_1)^2}{(1 + \delta_0\delta_1)^2 + (\delta_0 - \delta_1)^2} \quad |T|^2 = \frac{(\delta_0 - \delta_1)^2}{(1 + \delta_0\delta_1)^2 + (\delta_0 - \delta_1)^2} \tag{9}$$

As an example we assume $r = 1$ in (6). Then

$$\eta(x) = Z_0(\kappa(1 + \beta x/a)^{1/2}), \quad \kappa = 2k_1 a/\beta \tag{10}$$

and a particular combination of J_0, Y_0 which satisfies $\eta_0(0) = 0$, $\eta'_1(0) = 0$ is

$$\eta_i(x) = C_i(J_0(\kappa t)Y_i(\kappa) - Y_0(\kappa t)J_i(\kappa)), \quad (i = 0, 1), t = (1 + \beta x/a)^{1/2}, \tag{11}$$

Thus from (8) we obtain

$$\delta_i = -\mu \left(\frac{J_1(\kappa\alpha)Y_i(\kappa) - Y_1(\kappa\alpha)J_i(\kappa)}{J_0(\kappa\alpha)Y_i(\kappa) - Y_0(\kappa\alpha)J_i(\kappa)} \right), \quad \mu = \left(\frac{h_2}{h}\right)^{1/2} \alpha = (1+\beta)^{1/2} \tag{12}$$

As a check we let $\beta \to 0$ so that $h_2 \to h_1$ and we have a rectangular mount of width $2a$ submerged to a depth h_1 in water of depth h. Then $\mu \to (h_1/h)^{1/2}$, $\beta \to 0$ and $\kappa \to \infty$, $\kappa\alpha \sim \kappa + k_1 a$. Then using the results

$$J_i(z) \sim (\frac{2}{\pi z})^{1/2}\cos(z - \pi(1+2i)/4),\ Y_i(z) \sim (\frac{2}{\pi z})^{1/2}\sin(z - \pi(1+2i)/4),\ (i = 0, 1)$$

as $z \to \infty$, we obtain $\delta_0 \to \mu \cot k_1 a$, $\quad \delta_1 \to -\mu \tan k_1 a$, so that from (9)

$$|R|^2 = \frac{(1-\mu^2)^2 \sin^2 2k_1 a}{4\mu^2 + (1-\mu^2)^2 \sin^2 2k_1 a}, \quad |T|^2 = \frac{4\mu^2}{4\mu^2 + (1-\mu^2)^2 \sin^2 2k_1 a} \tag{13}$$

in agreement with Mei (1983) p.132.

A known solution of (1) is possible for $h^{(r)}(x)$ given by (5) with $r = 2$. Note that in this case α, β, κ, and μ are defined differently, so that for example $\alpha = (1+\beta) = (h_2/h_1)^{1/2}$. Let $(1 + \beta x/a) = e^t$. Then (1) becomes

$$\frac{d^2\eta(t)}{dt^2} + \frac{d\eta(t)}{dt} + \kappa^2\eta(t) = 0 \tag{14}$$

having the general solution $\eta(t) = e^{-t/2}(A\sin\lambda t + B\cos\lambda t)$ where $\lambda = (\kappa^2 - 1/4)^{1/2}$ and $\kappa = (k_1 a/\beta)$ and we assume $k_1 a > \beta/2$ since we wish to let $\beta \to 0$ later.
The odd and even solutions are given by

$$\eta_0(x) = C_0 e^{-t/2}\sin\lambda t, \quad \eta_1(x) = C_1 e^{-t/2}(2\lambda\cos\lambda t + \sin\lambda t) \tag{15}$$

satisfying $\eta_0(0) = 0$, $\eta_1'(0) = 0$ respectively, where $t = \log(1 + \beta x/a)$ so that $t = 0$ when $x = 0$ and $t = \log\alpha$ when $x = a$. Now

$$\eta_0'(x) = \beta a^{-1} e^{-t}\eta_0'(t) = C_0\beta a^{-1} e^{-3t/2}(\lambda\cos\lambda t - 1/2\sin\lambda t)$$

$$\eta_1'(x) = \beta a^{-1} e^{-t}\eta_1'(t) = -2C_1\beta a^{-1} e^{-3t/2}(\kappa^2\sin\lambda t)$$

We find from (8) that

$$\delta_0 = \gamma_0(\cot(\lambda\log\alpha) - 1/2\lambda), \quad \delta_1 = -\gamma_1/(\cot(\lambda\log\alpha) + 1/2\lambda) \tag{16}$$

where $\gamma_0 = \alpha\mu^2(\lambda\beta)/ka$, $\gamma_1 = \gamma_0(1 + 1/4\lambda^2) = \gamma_0\kappa^2/\lambda^2$, and $\mu = (h_1/h)^{1/2}$. Substitution in (9) gives

$$|R|^2 = \frac{A^2\sin^2(\lambda\log\alpha)}{4\gamma_0^2 + A^2\sin^2(\lambda\log\alpha)} \tag{17}$$

where $$A = 2(1 - \gamma_0\gamma_1)\cos(\lambda\log\alpha) + (1 + \gamma_0\gamma_1)\sin(\lambda\log\alpha)/\lambda.$$

As a check of (17) we let $\beta \to 0$ so that $h_2 \to h_1$ and as before we have a rectangular mount of width $2a$ submerged to a depth h_1 in water of depth h. Then and $\alpha \to 1$, whilst $\lambda\beta = ((k_1 a)^2 - \beta^2/4)^{1/2} \to k_1 a$, $\lambda\log\alpha \to k_1 a$ and $\lambda \to \infty$. Thus $\gamma_0 = \gamma_1 \to \mu$, and $A \to 2(1-\mu^2)\cos k_1 a$ so that $|R|^2$ given by (17) again is in agreement with (13). If in addition we let $h_1 = h_2 \to h$ so that the obstruction disappears, then $\mu \to 1$ and $A \to 0$ and hence $|R| \to 0$ as expected.
Notice that for the rectangular mount $R = 0$ when $\sin 2k_1 a = 0$ which Mei (1983) p.133 explains as contructive inteference between the waves generated at $x = \pm a$. It is not clear from (17) whether this explains in this case why $R = 0$ when $\sin\lambda\log\alpha = 0$ where $A \neq 0$, or when A=0.

A different scattering problem arises if we assume $h(x) = h_1$, $x \leq 0$ with a general $h(x)$ for $0 \leq x \leq a$ and $h(x) = h$, $x \geq a$ as before, so that we have the scattering problem

$$\eta(x) = e^{ik_1x} + Re^{-ik_1x},\ x \leq 0, \quad \eta(x) = Te^{ik(x-a)},\ x \geq a. \tag{18}$$

For $0 \leq x \leq a$ the equation (1) will, for a wide class of functions $h(x)$, have a general solution of the form $\eta(x) = A\eta_0(x) + B\eta_1(x)$ being a linear combination of two independent solutions which for ease of calculation we shall choose to be the functions $\eta_0(x)$ and $\eta_1(x)$ satisfying $\eta_0(0) = 0$ and $\eta_1'(0) = 0$ as before and which can be shown to satisfy the Wronskian-type relation

$$\eta_1(x)\eta_0'(x) - \eta_0(x)\eta_1'(x) = (h_1/h(x))\eta_1(0)\eta_0'(0). \tag{19}$$

To simplify the analysis, and without loss of generality, we also assume $\eta_0'(0) = \eta_1(0) = 1$. Then matching $\eta(x)$ and flux across $x = 0$ and $x = a$ gives

$$1 + R = B, \quad ik_1(1-R) = A, \quad T = A\eta_0(a) + B\eta_1(a), \quad ikhT = h_2(A\eta_0'(a) + B\eta_1'(a))$$

It follows after some algebra, which involves use of the relation (19) when $x = a$, that

$$R = (\nu_0 - i\nu_1/k_1)/(\nu_0 + i\nu_1/k_1), \quad T = 2(h/h_2)(\nu_0 + i\nu_1/k_1) \tag{20}$$

where $\nu_i = \eta_i'(a) - ik\eta_i(a)(h/h_2)$, $(i = 0, 1)$, and as before $\mu = (h_1/h)^{1/2}$. Further manipulation confirms that the energy relation $\mu(1 - |R|^2) = |T|^2$ is satisfied.
As a check on the result (20) we assume $h_2 = h_1$ so that we have a vertical step down at $x = a$ from depth h_1 to h. Then $\eta_0(x) = k_1^{-1}\sin k_1x$, $\eta_1(x) = \cos k_1x$, satisfying $\eta_0'(0) = \eta_1(0) = 1$, $\eta_0(0) = \eta_1'(0) = 0$, and we find after some algebra that

$$R = -\frac{(1-\mu)}{(1+\mu)}e^{2ik_1a}, \quad T = \frac{2}{1+\mu}e^{ik_1a}$$

in agreement with Mei (1983) p.119.

Finally we consider the scattering problem obtained by assuming that $h(x) = h_1$, $x \leq 0$ and that $h(x) \to \infty$, $x \geq 0$ noting that this would clearly violate shallow-water theory! The two solutions of (1), now valid in $x > 0$, need to be replaced by a single solution $\eta_{out}(x)$ describing a solution radiating outwards in $x > 0$. Simple matching gives

$$\frac{1+R}{1-R} = -\frac{ik_1\eta_{out}(0)}{\eta_{out}'(0)} \;=>\; R = -\left(\frac{\eta_{out}'(0) + ik\eta_{out}(0)}{\eta_{out}'(0) - ik\eta_{out}(0)}\right) \tag{21}$$

As an example with $h(x) = h_1(1 + \beta x/a)^r$, $x \geq 0$

$$\eta_{out}(x) = ((1 + \beta x/a)^{(1-r)/2} H_\nu^{(1)}(2k_1a(1 + \beta x/a)^{1-r/2}/\beta(2-r)), \tag{22}$$

for $0 < r \leq 1$, where $H_\nu^{(1)} = J_\nu + Y_\nu$, and if $r = 2$,

$$\eta_{out}(x) = (1 + \beta x/a)^{-1/2} e^{i\lambda \log(1+\beta x/a)}. \tag{23}$$

Conclusion

A solution of the linearised shallow-water equation for a more general bottom topography is given which is believed to be new. The solution provides the conditions for resonant oscillations in both a basin and a trough having curved sides. General expressions for the solution of a number of scattering problems are developed, mostly in terms of odd and even solutions to the shallow-water equation once they are known. These solutions are presented in a small number of cases which will enable reflection and transmission coefficients to be computed.

References

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