Singularity in the second order flexural-gravity waves

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Introduction

The research on the interaction of flexural-gravity waves with structures has received much attention in the past. Both the linear and nonlinear problems are of concern. The linear solutions to this problem have been developed using different analytic methods in both 2D and 3D formulations [1-5]. In the case of large-amplitude waves, the second-order solutions have also been proposed by BEM technique for a vertical circular cylinder surrounded by an infinite ice sheet [6, 7] and a flexible floating body with finite length [8, 9].

In the present work, the second order diffraction of a flexural-gravity wave by a vertical wall for finite water depth is investigated by the classical eigenfunction expansion method. A singularity of the second order incident potential has been discovered at a special wave frequency, where the wave speeds at this frequency and at the double frequency are equal. Vanden-Broeck and Parau [10] suggested a method to remove this singularity from strong nonlinear waves by adding a wave component at the double frequency. However, how to remove the singularity from the second order solution by regular perturbation analysis is still a problem.

Fundamental equation

As shown in Figure 1, the problem of the flexural-gravity waves propagating in a water of finite depth is considered first. A two-dimensional Cartesian coordinate system is used where the x-axis is horizontal and the z-axis is directed vertically upward. The upper surface of the fluid $-\infty < x < +\infty, z = 0$ is covered by a thin infinite elastic plate with a small thickness d, a Young's modulus E, a Poisson's ratio υ, and a density ρ_S. The rigid sea bed is flat, $z = -h$. The fluid is incompressible, inviscid, and the motion is irrotational. The nonlinearity of the problem comes from the Bernoulli equation for the hydrodynamic pressure and the kinematic condition on the elastic plate/water interface. The elastic deflections are linear and governed by the Euler-Bernoulli plate equation in this study.

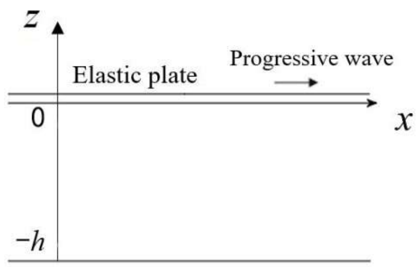

Figure 1: Flexural-gravity wave propagates in water of finite depth covered by infinite elastic plate

In the frequency domain, the dimensional velocity potential is expanded into a perturbation series,

$$\Phi = \mathrm{Re}\left[\varepsilon\phi^{(1)}e^{-i\omega t} + \varepsilon^2\phi^{(2)}e^{-2i\omega t}\right] + \varepsilon^2\phi_m^{(2)} + O(\varepsilon^3)\,. \tag{1}$$

As the mean term $\phi_m^{(2)}$ does not contribute to the wave force up to the second order, the stress will only be given on the oscillatory parts of the velocity potential, $\phi^{(j)}(x,z)\,(j=1,2)$. $\phi^{(j)}(x,z)$ satisfy Laplace's equation in the unperturbed fluid domain,

$$\partial^2\phi^{(j)} / \partial^2 x + \partial^2\phi^{(j)} / \partial^2 y = 0, \quad j = 1,2, \; (-h < y < 0)\,, \tag{2}$$

the sea-bed condition

$$\partial\phi^{(j)} / \partial z = 0, \qquad (z = -h)\,, \tag{3}$$

and the conditions on the upper boundary, z=0,

$$\left[EI\frac{\partial^4}{\partial x^4} - m_s(j\omega)^2 + \rho g\right]\phi_z^{(j)} - \rho(j\omega)^2\phi^{(j)} = F^{(j)}, \quad j = 1,2\,, \tag{4}$$

where ρ is the density of water and g is the acceleration due to gravity, $m_s = \rho_S d$ is density of the plate per unit area,

$I = d^3/12(1-\nu^2)$, and the forcing term reads

$$F^{(j)} = \begin{cases} 0, & j=1 \\ \dfrac{\mathrm{i}}{2\omega}[(EI\dfrac{\partial^4}{\partial x^4} - 4m_s\omega^2 + \rho g)\left(\phi_x^{(1)}\phi_{zx}^{(1)} - \phi_z^{(1)}\phi_{zz}^{(1)}\right)] + \dfrac{\mathrm{i}\omega\rho}{2}[\left(\phi_x^{(1)}\right)^2 + 3\left(\phi_z^{(1)}\right)^2], & j=2 \end{cases}. \tag{5}$$

Regular perturbation solution to the second order

The first order solution

For a propagating wave with an amplitude A,

$$\eta^{(1)} = Ae^{\mathrm{i}k_1 x} \quad , \tag{6}$$

the first order flexural-gravity velocity potential reads

$$\phi^{(1)} = -\frac{Ai\omega}{k_1 \tanh(k_1 h)} e^{ik_1 x} \frac{\cosh k_1(z+h)}{\cosh k_1 h}, \tag{7}$$

where k_1 is the wave number which satisfies the dispersion relation with the wave frequency ω,

$$\left[EIk_1^4 - m_s\omega^2 + \rho g\right]k_1 \tanh(k_1 h) - \rho\omega^2 = 0 . \tag{8}$$

The second order solution

Substituting the first order potential (7) into Eq. (5), the second order forcing term on the upper surface of the fluid can be written as

$$F^{(2)} = (\frac{A\omega}{k_1 \tanh(k_1 h)})^2[\frac{ik_1^3 \tanh(k_1 h)}{\omega}(16k_1^4 EI - 4m_s\omega^2 + \rho g) + \frac{i\rho\omega k_1^2}{2}(3\tanh^2(k_1 h) - 1)]e^{2ik_1 x} . \tag{9}$$

The second-order propagation potential can be derived as follow by the Laplace equation and the boundary conditions:

$$\phi^{(2)} = C\, e^{2\mathrm{i}k_1 x} \frac{\cosh 2k_1(z+h)}{\cosh 2k_1 h}, \tag{10}$$

where $C = F^{(2)}e^{-2\mathrm{i}k_1 x}/D$, and the denominator D is

$$D = (16k_1^4 EI - 4m_s\omega^2 + \rho g)2k_1 \tanh(2k_1 h) - 4\rho\omega^2 . \tag{11}$$

The singularity of the second order propagation potential

To show the character of the second order flexural-gravity velocity potential, we plotted out the variations of the coefficient *C* and denominator *D* with the wave frequency in Figs. 2 and 3, in which the model characteristic parameters are selected as follow: density of water $\rho = 1025kg/m^3$, density of plate $\rho_s = 925kg/m^3$, acceleration of gravity $g = 9.81m/s^2$, water depth $h = 30m$. The stiffness of the plate *EI* is selected as 1×10^7, 5×10^8 and $1\times10^{10}\,Nm$, respectively. For comparison, the ones of the gravity wave, or those with the free surface condition $(EI = \rho_s = 0)$, are also plotted and indicted as FS in the figures. It can be seen that for the cases with an elastic plate on the upper surface, there is a singularity for each case at the frequency corresponding to the zero of the denominator *D.* For the case with the free surface condition, there is no singularity in the whole frequency range as the denominator *D* does not have zero except at $\omega = 0$.

From further analysis it is found that the occurrence of *D=0* is due to the reason that k_2 , the wave number corresponding to the double wave frequency

$$\left[EI(k_2)^4 - m_s(2\omega)^2 + \rho g\right]k_2 \tanh(k_2 h) - \rho(2\omega)^2 = 0, \tag{12}$$

is equal to $2k_1$, twice of the wave number corresponding to the wave frequency. At this frequency, the speed of the locked second order wave $c_1 = 2\omega/2k_1 = \omega/k_1$ is equal to the speed of the free wave at the double frequency $c_2 = 2\omega/k_2$ $= \omega/k_1$. Fig. 4 shows the variation of the ratio of wave numbers $2k_1/k_2$ with the wave frequency. It is seen that for the cases with different elastic plates on the top of the water the ratios $2k_1/k_2$ are unit at zero frequency, and then decreases with the increase of wave frequency. After reaching its minimum, they begin to increase and cross the unit line. However, for the free surface condition, the ratio $2k_1 / k_2$ decreases monotonically with increase of the wave frequency and does not cross the unit line.

Fig. 5 shows the variation of the wave speed c with the wave frequency. For the cases with elastic plates on the top of the water the wave speed decreases with the increase of wave frequency firstly, and begins to increase after reaching its minimum. There is a possibility that $c(2\omega) = c(\omega)$ at some value of the wave frequency.

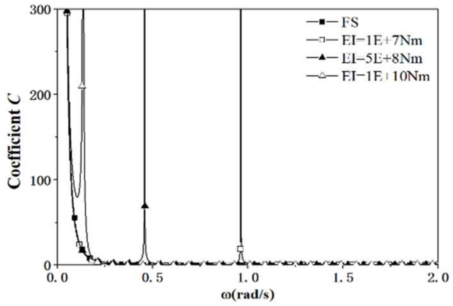

Figure 2: Variation of coefficient *C* with frequency

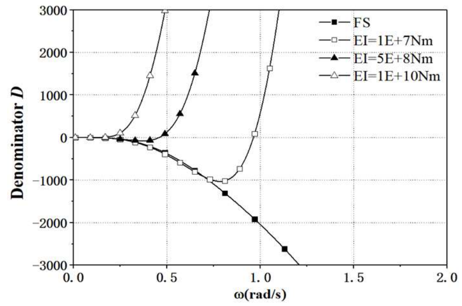

Figure 3: Variation of denominator *D* with frequency

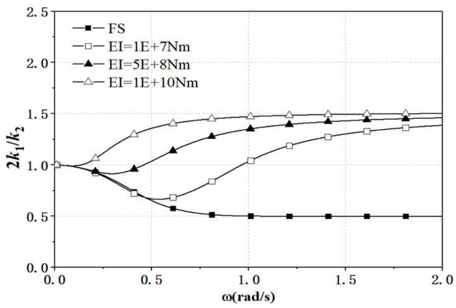

Figure 4: Variation of ratio $2k_1 / k_2$ with frequency

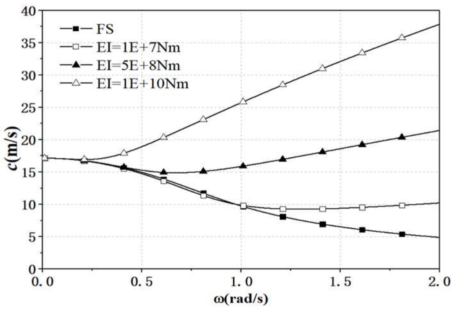

Figure 5: Variation of wave speed with frequency

Removing the singularity in the nonlinear flexural-gravity

Vanden-Broeck and Parau [10] studied the singularity problem in a moving coordinate system with the wave, and expanded the wave profile, potential function and wave speed into perturbation series to the third order by the singular perturbation method

$$\begin{aligned} \eta &= \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \varepsilon^3\eta^{(3)} + O(\varepsilon^4) \\ \phi &= \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \varepsilon^3\phi^{(3)} + O(\varepsilon^4) \\ c &= c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \varepsilon^3 c_3 + O(\varepsilon^4) \end{aligned} \tag{13}$$

For the infinite water depth situation, the singularity in the second order potential is removed by taking the first order flexural-gravity wave as:

$$\eta^{(1)} = \mathrm{A}_1 \cos kx \pm A_1 / 2 \cos 2kx \, . \tag{14}$$

They demonstrated that the solution is non-uniqueness.

Reflection of second order flexural-gravity wave from a vertical wall

To demonstrate the influence of the singularity in the second potential by regular perturbation method, a second order model is applied to the reflection of flexural-gravity waves from a vertical wall, as shown in Figure 6, with a second order incident wave. In the present numerical calculations, the clamped conditions $(\eta = \eta_x = 0,\ at\ x = 0)$ is applied at the contact of the elastic plate and the wall. The parameters of the problem are the same as above except the stiffness of the plate is $EI = 5\times10^8\,Nm$. The distribution of the module of the first order potential at the frequency $\omega = 1.0\ rad/s$ is shown in Figure 7. The second order force $F_x^{(2)}$ on the vertical wall is divided into two terms, the term due to the quadratic product of the first velocity

$$F_{x1} = -\frac{\rho}{4}\int_{-h}^{0} \nabla\phi^{(1)} \cdot \nabla\phi^{(1)} \mathrm{d}z \, , \tag{15}$$

and the term due to the second order potential

$$F_{x2} = 2\mathrm{i}\omega\rho\int_{-h}^{0}\phi^{(2)}\mathrm{d}z\ . \tag{16}$$

Figures 8 and 9 show that F_{x2} is much greater than F_{x1}. The component F_{x2} tends to infinity when the wave frequency approaches the special value where $2k_1 = k_2$. This is a problem for the second order analysis of the interaction of flexural-gravity with structures.

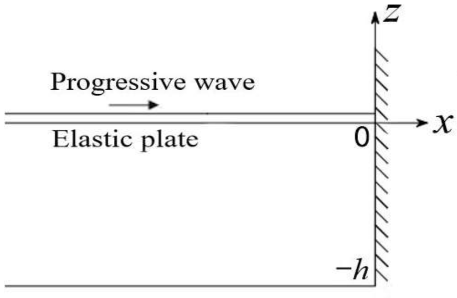

Figure 6: The interaction between flexural-gravity waves and a vertical wall with the clamped conditions.

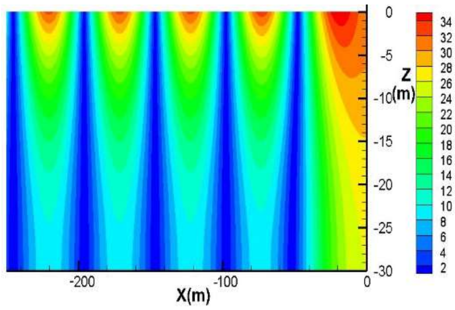

Figure 7: Module of the first order potential near the wall

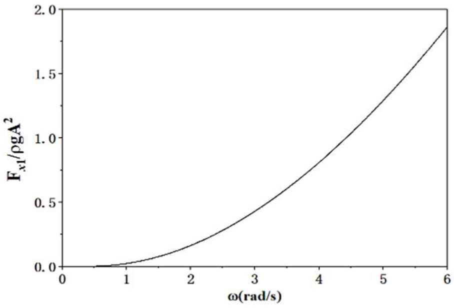

Figure 8: The force term F_{x1} on the vertical wall

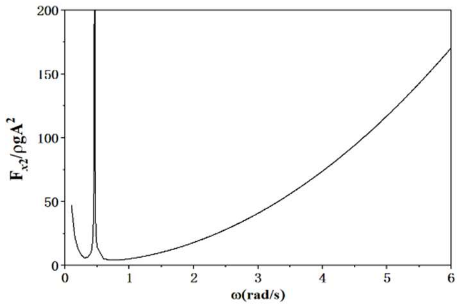

Figure 9: The force term F_{x2} on the vertical wall

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