On a new explicit time-integration method for advection equation and its application in hydrodynamics

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1 Introduction

The advection equation is a differential equation that governs the motion of a conserved scalar field as it is advected by a known velocity vector field. It has high relevance in solution of problems related to water waves and their interaction with fixed and floating structures. When the seakeeping problem with forward speed effects is considered in a reference frame moving with the ship speed, advective terms appear in both kinematic and dynamic free surface conditions [1] [2]. The advection terms are also an important part of the Navier-Stokes equations, representing great numerical challenges in terms of both accuracy and stability for flows dominated by advection. There exist many numerical methods that can be used to solve the advection equation in the time domain. Examples are the one-stage explict and implict Euler methods, as well as multi-stage methods which can be built from the one-stage methods.

Explicit methods are normally easy to implement and cheaper to solve within one time step for a given spatial discretization. However, they normally require small time steps to achieve stable solution if possible. The stability of the explicit methods is strongly affected by the numerical scheme for the spatial discretization. It can be shown by a von Neumann stability analysis that, if a scheme based on forward in time and central difference in spatial derivatives is used in solving a periodic problem, the solution is unconditionally unstable. Therefore, upwind finite difference schemes are often applied to stabilize the solution, and the Courant–Friedrichs–Lewy (CFL) number must be smaller enough.

We present a new class of explicit scheme which is derived from an approximation of the implicit Euler scheme. This scheme does not involve solving matrix equations that are required by a standard implicit Euler scheme. Unlike the standard explicit Euler scheme, which is unconditionally unstable for central difference schemes in spatial descretization, the proposed scheme is conditionally stable for any types of differential operators in the advection terms. The linear stability analysis results and two examples of the application of the new explicit schemes will also be shown.

2 An explicit approximation of the implicit Euler method

Taking the following one-dimensional advection equation as example

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0, \tag{1}$$

and applying the standard implicit Euler method, one can formally write the discretization of Eq. 1 as

$$\frac{\vec{\phi}^{n+1} - \vec{\phi}^{n}}{\Delta t} + u\mathbf{D_x}\vec{\phi}^{n+1} = 0, \quad i = 1, ..., N, \tag{2}$$

where the advection term is discretized using the solution at current time step, i.e. $\vec{\phi}^{n+1}$. In a standard one-stage explicit method, the advection term will be approximated using solutions at previous time step i.e. $\vec{\phi}^{n}$. N is the total number of grid points. $\mathbf{D_x}$ is the operator for the spatial differentiation $\frac{\partial}{\partial x}$, based on standard finite difference schemes or any other velocity reconstruction methods using stencil points around the grid points of interest.

It should be noted that $\mathbf{D_x}$ is dependent on the size of meshes, thus we can introduce characteristic mesh size Δs_c at each grid point and rewrite Eq. 2 as

$$\vec{\phi}^{n+1} + r\tilde{\mathbf{D}}_\mathbf{x}\vec{\phi}^{n+1} = \vec{\phi}^{n}, \quad i = 1, ..., N, \tag{3}$$

where we have defined

$$r = \frac{u\Delta t}{\Delta s_c}, \tag{4}$$

and

$$\tilde{\mathbf{D}}_\mathbf{x} = \Delta s_c \mathbf{D_x}. \tag{5}$$

Here r is proposional to the CFL number. The scaled operator $\tilde{\mathbf{D}}_\mathbf{x}$ has less dependence on grid size than the un-scaled operator $\mathbf{D_x}$. Taking a uniform grid and applying the popular 2nd order central difference as an example, the i-th component of Eq.3 becomes

$$r\phi_{i+1}^{n+1} + \phi_i^{n+1} - r\phi_{i-1}^{n+1} = \phi_i^{n}, \tag{6}$$

where $r = u\Delta t/2\Delta x$. Δx is the grid size. The subscript and superscript indicate the index of grid point and time step number, respectively.

After considering proper boundary conditions at two ends of the domain, we can formally set up a matrix equation for the solution at n-th time step and the boundary conditions.

$$(\mathbf{I} + r\tilde{\mathbf{D}}_{\mathbf{x}})\vec{\phi}^{n+1} = \vec{\phi}^{n}. \tag{7}$$

Here $\mathbf{I}$ is identity matrix. $\mathbf{A} = \mathbf{I} + r\tilde{\mathbf{D}}_{\mathbf{x}}$ is a sparse matrix whose bandwidth is equal to the size of local stencil points.

The standard way of solving Eq.7 is to solve the sparse-matrix equation iteratively, which may be computationally costly as the matrix on the left-hand side is not diagonal-dominiant. If we consider r as a small parameter, e.g. $r \ll 1$, we can approximate the inverse of $\mathbf{A} = \mathbf{I} + r\tilde{\mathbf{D}}_{\mathbf{x}}$ by series expansion in r

$$\mathbf{A}^{-1} = \mathbf{I} + \sum_{k=1}^{\infty} \left(r^k \mathbf{C}_k\right) \tag{8}$$

Here the matrix $\mathbf{C}_k$ with $k = 1, ..., \infty$ are independent on r. To determine $\mathbf{C}_k$, we take the product of $\mathbf{A}$ and $\mathbf{A}^{-1}$:

$$\mathbf{A}\mathbf{A}^{-1} = \left[\mathbf{I} + r\tilde{\mathbf{D}}_{\mathbf{x}}\right]\left(\mathbf{I} + \sum_{k=1}^{\infty} r^k \mathbf{C}_k\right) \equiv \mathbf{I} \tag{9}$$

Eq.9 can be further re-arranged by collecting the terms with the same order in r as

$$r^1\left(\tilde{\mathbf{D}}_{\mathbf{x}} + \mathbf{C}_1\right) + r^2\left(\tilde{\mathbf{D}}_{\mathbf{x}}\mathbf{C}_1 + \mathbf{C}_2\right) + \cdots + r^k\left(\tilde{\mathbf{D}}_{\mathbf{x}}\mathbf{C}_{k-1} + \mathbf{C}_k\right) + \cdots = \mathbf{0} \tag{10}$$

Requiring each term in the Eq.10 to vanish leads a simple solution for $\mathbf{C}_k$:

$$\mathbf{C}_k = (-1)^k \tilde{\mathbf{D}}_{\mathbf{x}}^k, \quad k = 1, \cdots, \infty \tag{11}$$

Therefore, $\mathbf{A}^{-1}$ in Eq.8 can be approximated in explicit form as

$$\mathbf{A}^{-1} = \mathbf{I} + \sum_{k=1}^{N_p} (-r)^k \tilde{\mathbf{D}}_{\mathbf{x}}^k + O(r^{N_p+1}), \tag{12}$$

where the we have truncated the infinite series summation to N_p terms.

To this end, we can approximate the implicit Euler scheme as

$$\frac{\vec{\phi}^{n+1} - \vec{\phi}^{n}}{\Delta t} = \left[\sum_{k=0}^{N_p-1} (-1)^{k+1} (\Delta t)^k \left(u\mathbf{D}_{\mathbf{x}}\right)^{k+1}\right] \vec{\phi}^{n} + O(\Delta t), \tag{13}$$

which is an explicit scheme and conditionally stable as we will show later in the stability analysis.

When $N_p = 1$, the scheme in Eq.13 becomes exactly the same as explicit Euler method. When $N_p > 1$ is considered in the truncated summation, the additional terms will provide stability of the scheme even for central difference schemes. Extension to higher-order schemes and other properties of the scheme and its generalization will also be presented at the workshop.

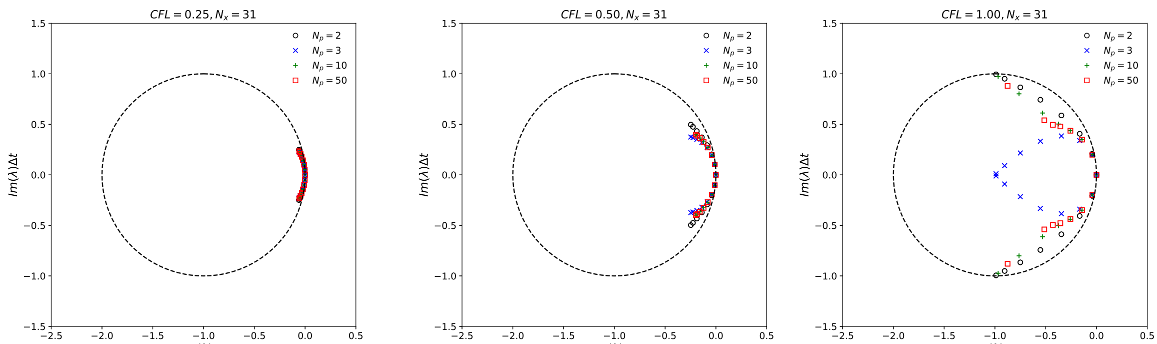

Figure 1: Results of linear stability analysis for the new explicit scheme for different CFL numbers. Results are based on 2nd order central difference for $\mathbf{D}_{\mathbf{x}}$. N_p is the total number of terms in the series expansion in Eq.13

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3 Linear stability analysis

By taking 2nd order central difference as an exmaple for the calculation of $\tilde{\mathbf{D}}_{\mathbf{x}}$, we have carried out the linear stability analysis for the scheme in Eq.13. A method of lines approach [3] which allows for separate consideration of the time integration scheme and the spatial discretization is applied. Fig.1 shows the linear stability analysis results for the new explicit scheme derived from the approximated implicit Euler method. Results are shown for CFL = 0.25, 0.5 and 1.0, as well as different number of terms N_p in the exansion in Eq. 13. For small CFL numbers, the eigvenvalues of the Jacobian matrix are less dependent on N_p, while they tend to be different for larger CFL numbers. It indicates that the importance of the higher order terms in the expansion increases for increasing CFL number.

4 Applications in hydrodynamics

The method presented above is being applied in a weakly-nonlinear seakeeping solver based on perturbation scheme and a Navier-Stokes equation solver. For the seakeeping problem, we will present the wave diffraction problem of a vertical circular cylinder under forced low-frequency surge motion in regular waves. This problem has relevance for moored marine structures under large low-frequency motions in the horizontal plane. For simplicity, we take a cylinder with a draft equal to the water depth.

The problem is formulated in a non-inertial body-fixed coordinate system , and the boundary value problem is solved by a time-domain higher-order boundary element method [1] [4]. For a k-th order problem (k=1, 2), the dynamic and kinematic free surface conditions take the following form

$$\left(\frac{\partial}{\partial t} - \left(\vec{W}^0 - \nabla\phi^0\right)\right)\phi^{(k)} = -g\eta^{(k)} + f_1^{(k)} \quad on\ z=0,\ k=1,2, \tag{14}$$

$$\left(\frac{\partial}{\partial t} - \left(\vec{W}^0 - \nabla\phi^0\right)\right)\eta^{(k)} = \frac{\partial\phi^{(k)}}{\partial z} + f_2^{(k)} \quad on\ z=0,\ k=1,2. \tag{15}$$

The $-\left(\vec{W}^0 - \nabla\phi^0\right)$ term, which is due to the steady or slowly-varying velocity of the structure, can be considered as advection velocity. The superscript k stands for the order of the hydrodynamic problem. $f_1^{(k)}$ and $f_2^{(k)}$ are the forcing terms which are considered as known. A 2nd order time-integration scheme developed from an approximation of the implicit Crank-Nicolson method is applied for the time-stepping of the free surface conditions. In Fig.2, the 1st and 2nd order components of diffraction force on the vertical circular cylinder undergoing a slowly-varying surge oscillation are presented. A regular incident wave with wave kR=1.0 is considered, with R as the cylinder radius and k as the wave number. The slowly-varying surge velocity of the cylinder is $U_1 = U_a\cos(0.1\omega t)$, where ω is the incident wave frequency. U_a is the surge velocity amplitude we have used $U_a/\sqrt{gR} = 0.16$ in the analysis. Large influence of slowly-varying motions on the 2nd order wave loads is seen, while their effects on the 1st order wave loads are relatively smaller.

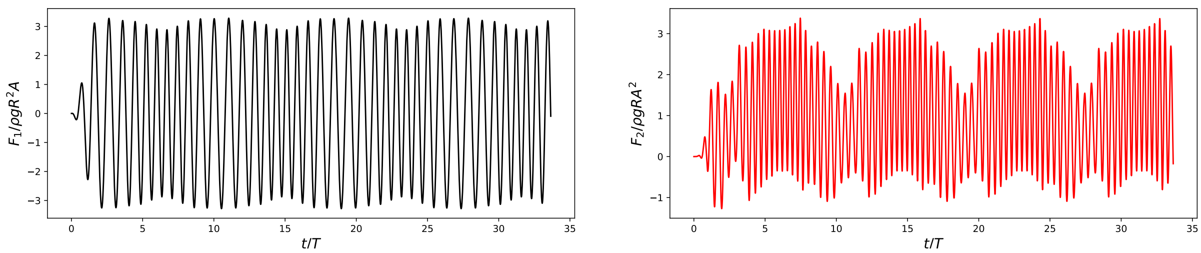

Figure 2: Non-dimensional 1st order and 2nd order diffraction forces on vertical circular cylinder under forced low-frequency oscillations. Left: 1st order. Right: 2nd order.

As the second example, we will present our our attempt to apply the new time-integration scheme in solving the 2D Navier-Stokes equations. The following 4-step projection method is applied:

$$\frac{\vec{u}^* - \vec{u}^n}{\Delta t} + (\vec{u}^n\cdot\nabla)\vec{u}^* = 0 \tag{16}$$

$$\frac{\vec{u}^{**} - \vec{u}^n}{\Delta t} + \nu\nabla^2\vec{u}^{**} = 0 \tag{17}$$

$$\nabla^2 p^{**} = \frac{\rho}{\Delta t}\nabla\cdot\vec{u}^{**} \tag{18}$$

$$\frac{\vec{u}^{n+1} - \vec{u}^{**}}{\Delta t} = \frac{\nabla p^{**}}{\rho}. \tag{19}$$

In order to test the accuracy and stability of our explicit scheme for advection equation, we have decided to solve the diffusion equation by an implicit Euler scheme, thus the stability of the solution will be dominated by the advection step. A 'frozen nonlinearity' approach has been used for the advection terms, i.e. the advection velocity will be approximated by the previous time step. Alternatively, one can use, for instance, a Adams-Bathforth predictor to estimate the advection velocity from previous steps. The advection step is then solved by the developed explicit scheme which is an approximation of the implicit Euler method. All spatial derivatives of velocities and pressure are calculated by the 2nd order central difference. The steady Kovasznay flow is considered here to test the stability of the scheme. The exact solution for the velocity and pressure fields is given by

$$u = 1 - e^{\lambda x} cos(2\pi y), \quad v = \frac{\lambda}{2\pi} e^{\lambda x} sin(2\pi y), \quad p = \frac{1}{2}\left(1 - e^{2\lambda x}\right), \tag{20}$$

where $\lambda = Re/2 - \sqrt{Re^2/4 + 4\pi^2}$. Re is the Reynolds number.

Fig.3 shows the computed streamlines for the flow field over a [-0.5, 1.0] × [-0.5, 0.5] computational domain, which is descretized by a 80 x 60 uniform grid. The stability of the new explicit scheme approximated from the standard implicit Euler scheme is compared with the explicit Euler scheme in Fig.4 for $Re = 500$. Time series of L_2 norm if the calculated horizontal velocity are presented for both standard explicit Euler method and the new explicit method. Different curves represent results for different time-step sizes. It is seen from the comparison that much larger time steps can be used in the new time-integration method. More results for even higher Reynolds number will be presented at the workshop.

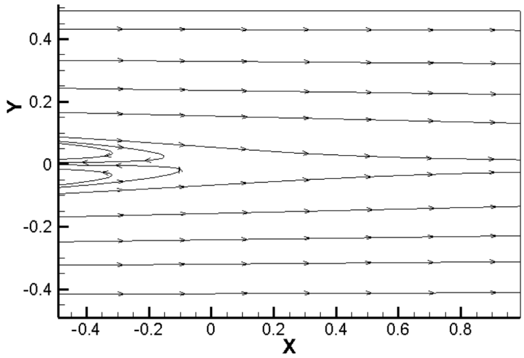

Figure 3: Stream line of the steady Kovasznay flow in a [-0.5, 1.0] × [-0.5, 0.5] computational domain

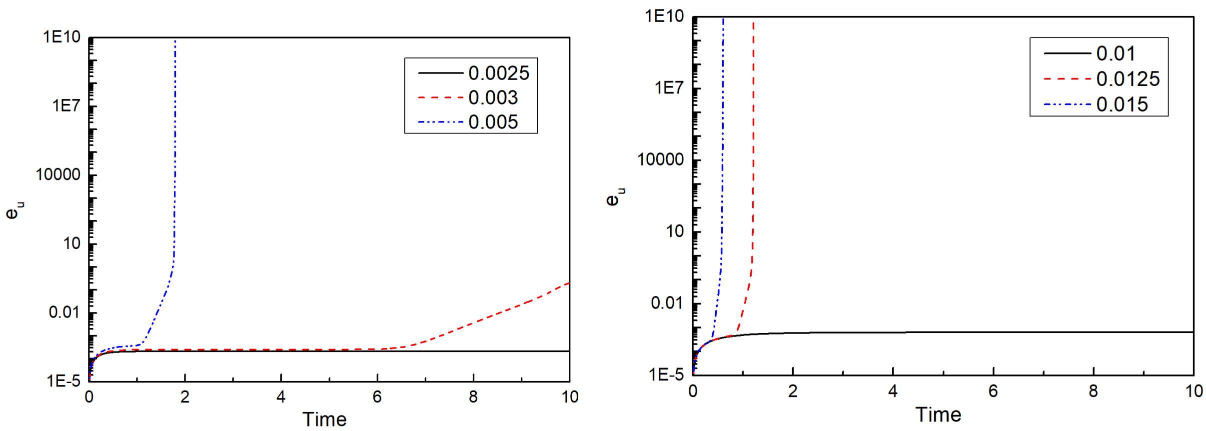

Figure 4: Time series of the x-component velocity L_2 norm with different step size using different schemes for the advection step. Different curves represent result for different time step. Left: standard Euler scheme; Right: the new explicit scheme approximated from implicit Euler scheme

References

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