Flexural Gravity Wave Scattering for Compressed Ice

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1 Introduction

The problem of scattering for flexural-gravity waves has been the subject of extensive research motivated by understanding the wave processes which are prominent in the ice-infested ocean and the engineering challenges associated with very large floating structures. Many different problems in wave scattering have been solved, and the field has reached a stage of maturity ?Squire (2007). However, the inclusion of compression, especially powerful compression or other effects such as a current causes the wave propagation behaviour to change Davys *et al.* (1985); Das *et al.* (2018*a*,*b*). In particular, if we consider the case of compression only, when the compression is sufficiently large (but below the buckling limit) there exists three travelling waves. For one of these waves the phase and group velocity are in opposite directions. We also have frequencies for which wave blocking can occur. These two effects greatly complicate some of our traditional notations of energy conservation and scattering.

2 The Governing Equations

The equations of motion for a Kirchhoff—Love theory plate including compression is given by

$$D\Delta^2\zeta + Q\Delta\zeta = -P(x,y,t) - \rho_i\tau\,\frac{\partial^2\zeta}{\partial t^2},$$

where ζ is the displacement of the plate, D is the flexural rigidity, Q is the compressive force. ρ_i is the density of the plate, τ is the plate thickness, and P is the pressure. At this point, we note that an infinite plate is unstable for any compression. However, this behaviour is altered by the presence of a fluid layer. We consider the problem of a Kirchhoff—Love theory thin plate which floats on the surface of a fluid. The governing equations (which are well known, e.g.) are shown in Figure 1. We note that these equations have been applied widely in polar and offshore engineering.

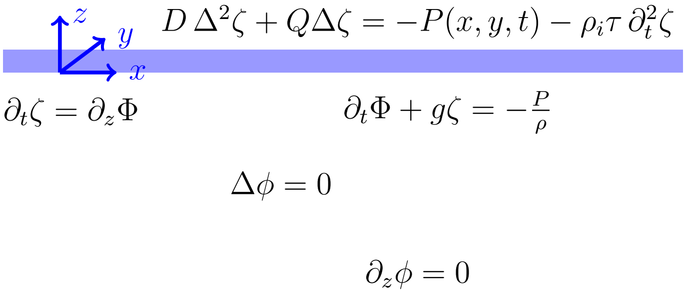

Figure 1: Basic Equations of Linear Flexural Gravity Waves with Compression

2.1 Plane Progressive Waves

The free surface elevation for a plane progressive wave is given by

$$\zeta(x,t) = \mathrm{Re}\{Ae^{\mathrm{i}kx-\mathrm{i}\omega t}\}, \tag{1}$$

where A is an a priori known complex wave amplitude The resulting plane progressive wave velocity potential takes the form

$$\phi(x,z,t) = \mathrm{Re}\left\{A\frac{ig}{\omega}\frac{\cosh k(z+h)}{\cosh kh}e^{\mathrm{i}kx-\mathrm{i}\omega t}\right\}, \tag{2}$$

where A is the amplitude in displacement and the dispersion relation is given by

$$k\tanh(kh)\left(Dk^4 - Qk^2 + \rho g - \omega^2\tau\rho_i\right) = \rho\omega^2. \tag{3}$$

For a typical problem we have

$$1 >> \omega^2\tau\rho_i,$$

and it is sensible to ignore the inertia effects of the plate. Including this effect generally does not make a large difference. We can write $D' = D/(\rho g)$, $Q' = Q/(\rho g)$ and it becomes

$$k\tanh(kh)\left(D'k^4 - Q'k^2 + 1\right) = \frac{\omega^2}{g}. \tag{4}$$

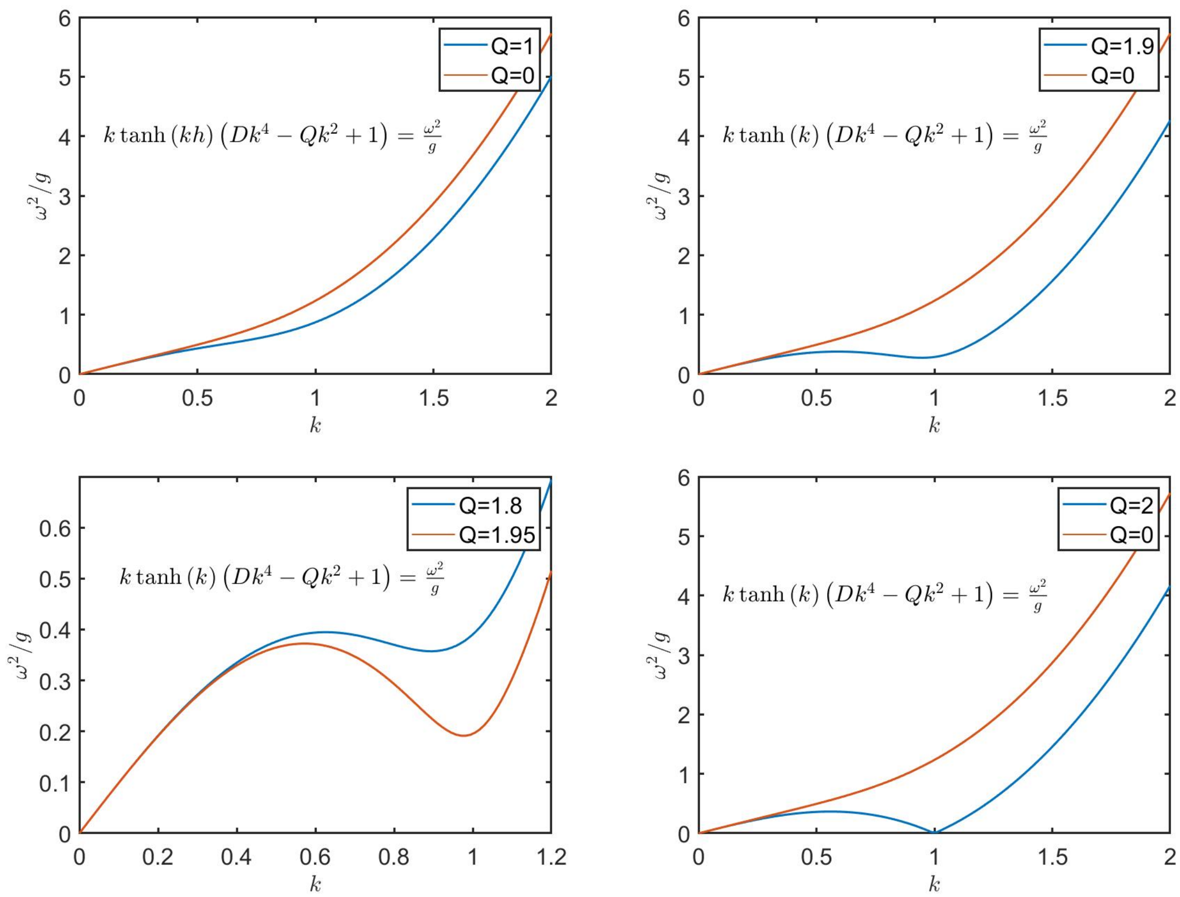

Figure 2: Effect of Q on the dispersion equation assuming that $D = h = 1$.

We will drop the prime from now on for simplicity. The dispersion curves for a range of Q values are shown in Figure 2. This figure illustrates that, once the compression becomes sufficiently large, there exist three distinct roots of the dispersion equation for certain values of ω. The middle value of k has a negative slope which means that the group and phase velocity are pointing in different directions. The origin of these extra solutions is the complex roots of the dispersion equation which occur when$Q = 0$. As the compression is increased, these complex roots become only real, and at the critical point where the roots first become real, there is a double root on the real axis. Note that $Q = 2$ (for $D = 1$) is the critical compression at which point the bending state (associated with $k = 1$) has lower energy than the plate in compression, i.e. the increase in gravitational potential energy is balanced by the decrease in energy of the plate due to the compression. This point is known as the buckling limit.

3 Scattering

We present here some simple scattering problems, focusing on the case when there are three travelling roots. In this case the effect of scattering can be to transform the wave in one incident wavenumber to a different wavenumber. We consider here as illustrative example, the scattering by a step. The depth is given by h for $x < 0$ and d for $x > 0$. We write the solution in imaginary form as

$$k \tan(kh)\left(D'k^4 - Q'k^2 + 1\right) = -\frac{\omega^2}{g} \text{ and } \kappa \tan(\kappa d)\left(D'\kappa^4 - Q'\kappa^2 + 1\right) = -\frac{\omega^2}{g}. \tag{5}$$

We can expand the solution in the following eigenfunction expansion

$$\phi_m(z) = \frac{\cos k_m(z+h)}{\cos k_m h}, \quad m \geq -2, \ x < 0, \tag{6}$$

and

$$\psi_m(z) = \frac{\cos \kappa_m(z+d)}{\cos \kappa_m d}, \quad m \geq -2, \ x < 0. \tag{7}$$

$$\phi = \begin{cases} \dfrac{A}{\mathrm{i}\sqrt{\omega}} \mathrm{e}^{\mathrm{i}k_j x}\phi_i(z) + \displaystyle\sum_{m=0}^{N} a_m \mathrm{e}^{-\mathrm{i}k_m x}\phi_m(z), \ x < 0, \\ \displaystyle\sum_{m=-2}^{N} b_m \mathrm{e}^{\mathrm{i}\kappa_m x}\psi_m(z), \ x > 0, \end{cases} \tag{8}$$

where A is the incident wave amplitude. In this expression we have explicitly defined the sign of the solutions of the dispersion equation (2) as follows. For the case of only one real solution, k_{-1} and k_{-2} are the complex solutions with positive imaginary part. In this case only wave of number $j = 0$ can be incident. In the case when there are three real roots the middle root must be negative (due to the negative group velocity). The case of repeated roots does not occur because in this case we take the real root which is the continuation of the complex root from the opposite side. The case of three coincident roots does occur,but only for a critical value of Q and ω .

The unknowns a_n and b_n are solved for by straight forward eigenfunction matching. The solution for scattering by a finite step can be calculated by using a decomposition using symmetry will minimal effort from this solution. The equations for this case are not given here due to a constraint of space.

3.1 Three–dimensional solution

The solution for scattering by a circular step can be found almost trivially from the two–dimensional solution. We assume that the step has radius a. Therefore the potential can be expanded as:

$$\phi(r,\theta,z) = \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty}\sum_{m=-2}^{\infty} a_{mn} K_n(k_m r)\mathrm{e}^{\mathrm{i}n\theta}\phi_m(z), \ r > a. \\ \displaystyle\sum_{n=-\infty}^{\infty}\sum_{m=-2}^{\infty} b_{mn} I_n(\kappa_m r)\mathrm{e}^{\mathrm{i}n\theta}\psi_m(z), \ r < a, \end{cases} \tag{9}$$

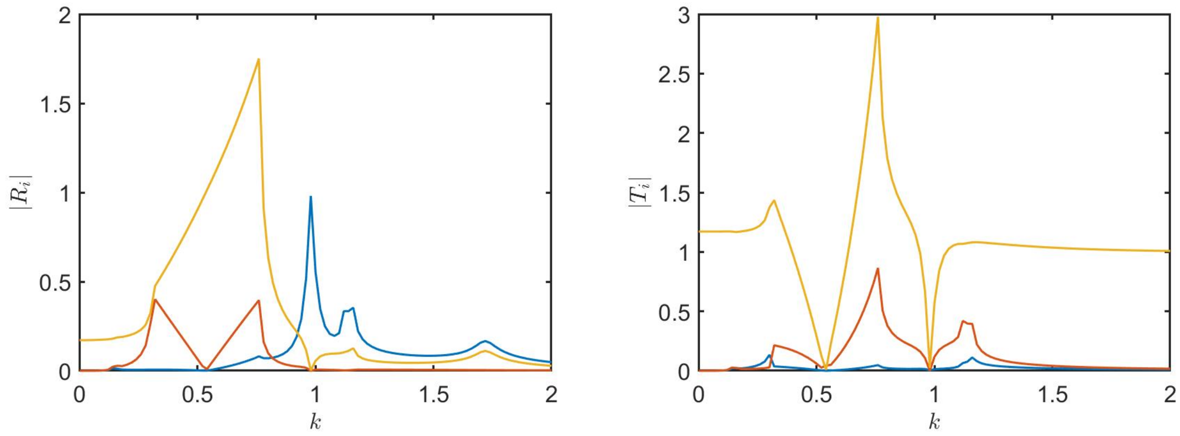

Figure 3: The reflection coefficients and transmission coefficients for $Q = 1.95$, $h = 1$ and $d = 2$.

plus the incident wave.

4 Results

We show here some fundamental results (Figure 3). More detailed examples including time–domain simulations will be shown in the workshop. For each value of k there is only a single solution, whereas for ω the solution can be multiply valued. We therefore fix the incident wavenumber, and we consider the solution for the three reflection and transmission modes, R_i and T_i respectively. The sharp jumps in the transmission occur at the points of blocking where the group velocity is zero. Mode conversion can also be observed, and we can see that there are points where the reflected and transmission amplitude is greater than unity. However, energy is still conserved because of the decrease in group velocity.

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