(*Abstract for the 35th International Workshop on Water Waves and Floating Bodies, Seoul, Korea, 2020*)

# A modified Benjamin-Feir index for crossing sea states


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## 1 Introduction

The wave systems characterized by two different spectral peaks with different propagation directions, also known as crossing sea states, have been regarded as a situation that may increase the effects of modulational instability. The nonlinear interactions of two wave systems appear to be a cause of the rogue wave occurrences. Although most of the available studies have focused on sea states characterized by single-peaked spectra, there is a significant percentage of sea states that are more complex than single-peaked spectral seas. There are many evidences that unusual and extreme wave phenomenons do occur in crossing seas. The famous Draupner wave is found to be generated in crossing seas according to the hindcast date from the European Centre for Medium-Range Weather Forecasts (Adcock *et al.*, 2011).

Here, we shall limit attention to the crossing seas consisted of two waves with nearly the same peak frequency. For nearly unidirectional crossing seas consisted of two identical wave systems, the modulational instability has been investigated with the framework of nonlinear Schrodinger (NLS) equations. Onorato *et al.* (2006) studied the modulational instability of two wave systems with the same frequency but different propagation direction using coupled nonlinear Schrodinger (CNLS) equations derived from Zakharov equation. Their results suggested that the existence of another wave system can result in an increase of the instability growth rates and the enlargement of instability region, and more rogue waves are expected in crossing sea states with crossing angles less than about 70°. Extension of the obtained results in Onorato *et al.* (2006) to the more general cases of two-dimensional perturbations has been performed b Shukla *et al.* (2006). Onorato *et al.* (2010) presented a further detailed derivation of the CNLS equations and a discussion of the coefficients in front of the dispersive and nonlinear terms with the support of direct numerical simulations. The result indicated that crossing angles between 20° and 60° are the most probable for establishing a freak wave sea. These theoretical results based on CNLS equation have been validated by Toffoli *et al.* (2011), who performed both laboratory experiments and numerical simulations to study the effect of crossing angle on the extreme events. Based on hindcast data and numerical simulations, Bitner-Gregersen & Toffoli (2014) found that the maximum kurtosis occurs for the crossing angle about 40°, independent on the wave directional spreading. A more recent experimental study reported in Luxmoore *et al.* (2019) shows that the third-order nonlinearity was more affected by varying the directional spreading of the components instead of the crossing angles between components. They also found that the kurtosis, which quantitatively describes the third-order nonlinearity, can be estimated quite well from the directional spreading using an empirical relationship based on the two-dimensional Benjamin-Feir index ($\mathrm{BFI_{2d}}$), proposed by Mori *et al.* (2011).

In the present paper, we derived a modified coupled two-dimensional Benjamin-Feir index ($\mathrm{CBFI_{2d}}$) for crossing seas to estimate the third-order nonlinearity effects. The prediction from $\mathrm{CBFI_{2d}}$ was validated based on numerical simulation using a higher-order spectral method as well as previous experimental data.

## 2 Derivation of a modified Benjamin-Feir index for crossing seas

Benjamin-Feir index (BFI) was an measurement of the importance of modulational instability for For narrow-banded long-crested waves (see Janssen, 2003). Mori *et al.* (2011) suggested an extension of the BFI, the two-dimensional Benjamin-Feir index ($\mathrm{BFI_{2d}}$), for waves with significant directional spread. A robust relation between $\mathrm{BFI/BFI_{2d}}$ and the kurtosis was built up for both long-crested and short-crested seas (Mori & Janssen, 2006; Mori *et al.*, 2011; Luxmoore *et al.*, 2019):

$$\mathrm{Kur} \quad = \quad \frac{\pi}{\sqrt{3}}\mathrm{BFI}^2_{(2d)} + 24\varepsilon^2 + 3, \tag{1}$$

Where Kur is the kurtosis value and $\varepsilon$ is wave steepness.

To describe the evolutions of two crossing waves with identical and symmetrical wavenumber, the coupled nonlinear Schrodinger (CNLS) equations have been derived from Zakharov equation based under the assumption

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that both wave systems are narrow-banded in Onorato *et al.* (2006). Considering the stability analysis of perturbations along $x$ axis, in a frame of reference moving with the group velocity the CNLS equations are given by:

$$\frac{\partial A}{\partial t} - i\alpha \frac{\partial^2 A}{\partial x^2} + i\left(\xi|A|^2 + 2\zeta|B|^2\right) A = 0, \tag{2}$$

$$\frac{\partial B}{\partial t} - i\alpha \frac{\partial^2 B}{\partial x^2} + i\left(\xi|B|^2 + 2\zeta|A|^2\right) B = 0, \tag{3}$$

where $A$ and $B$ are the complex amplitudes for two wave system, respectively. The corresponding wavenumber are $\mathbf{k_A} = (k, l)$, $\mathbf{k_B} = (k, -l)$, symmetrically propagating about the x-axis at angle $\pm\theta$. The coefficients in CNLS equations are defined as follows:

$$\alpha = \frac{\omega(\kappa)}{8\kappa^4}\left(2l^2 - k^2\right), \tag{4}$$

$$\xi = \frac{1}{2}\omega(\kappa)\kappa^2, \tag{5}$$

$$\zeta = \frac{\omega(\kappa)}{2\kappa}\left(\frac{k^5 - k^3 l^2 - 3kl^4 - 2k^4\kappa + 2k^2 l^2\kappa + 2l^4\kappa}{-2k^2 - 2l^2 + k\kappa}\right), \tag{6}$$

where $\kappa = \sqrt{k^2 + l^2}$, and $\omega$ is the corresponding angular frequency. To investigate analytically the crossing system further, we make the hypothesis that the evolutions of two envelope $A$ and $B$ are the same. The equations (2) and (3) are reduced to:

$$\frac{\partial A}{\partial t} + i\frac{1}{8}\frac{\omega(\kappa)}{\kappa^2}\beta\frac{\partial^2 A}{\partial x^2} + i\frac{1}{2}\omega(\kappa)\kappa^2(1+\gamma)A|A|^2 = 0. \tag{7}$$

The new coefficients are given by:

$$\beta = k^2 - 2l^2, \tag{8}$$

$$\gamma = \frac{2k^5 - 2k^3 l^2 - 6kl^4 - 4k^4\kappa + 4k^2 l^2\kappa + 4l^4\kappa}{(k - 2\kappa)\kappa}. \tag{9}$$

Following Serio *et al.* (2005), we rewrite the equation in non-dimensional form by introducing the following non-dimensional quantities:

$$A' = \frac{A}{\sqrt{2}a}, \quad x' = \Delta k x, \quad t' = \frac{\omega(\kappa)\Delta k^2 \beta}{8\kappa^2}t, \tag{10}$$

where $\Delta k$ denotes the spectral bandwidth and $a$ corresponds to the wave amplitude. The non-dimensional CNLS equations become (the primes have been now omitted for brevity):

$$\frac{\partial A}{\partial t} + i\frac{\partial^2 A}{\partial x^2} + i\left(\frac{2\sqrt{2}\kappa a}{\Delta k/\kappa}\right)^2 \frac{\gamma+1}{\beta}A|A|^2 = 0. \tag{11}$$

Based on the ratio of the nonlinear and dispersive term, we introduce a coupled Benjamin-Feir index (CBFI) for crossing sea states:

$$\mathrm{CBFI} = \frac{2\sqrt{2}\kappa a}{\Delta k/\kappa}\sqrt{\frac{\gamma+1}{\beta}}. \tag{12}$$

Figure 1 illustrates the variation of the coefficients as a function of propagation direction $\theta$. Note that the crossing angle is equivalent to $2\theta$. Considering the definition of BFI, this equation is rewritten as:

$$\mathrm{CBFI} = \mathrm{BFI}\sqrt{\frac{\gamma+1}{\beta}}. \tag{13}$$

For more general two-dimensional cases, we obtained:

$$\mathrm{CBFI_{2d}} = \mathrm{BFI_{2d}}\sqrt{\frac{\gamma+1}{\beta}}, \tag{14}$$

The $\mathrm{CBFI_{2d}}$ allows to evaluate the kurtosis value which is regarded as an important indicator of rogue wave occurrence, based on the linear relationship between kurtosis value and squared Benjamin-Feir index.

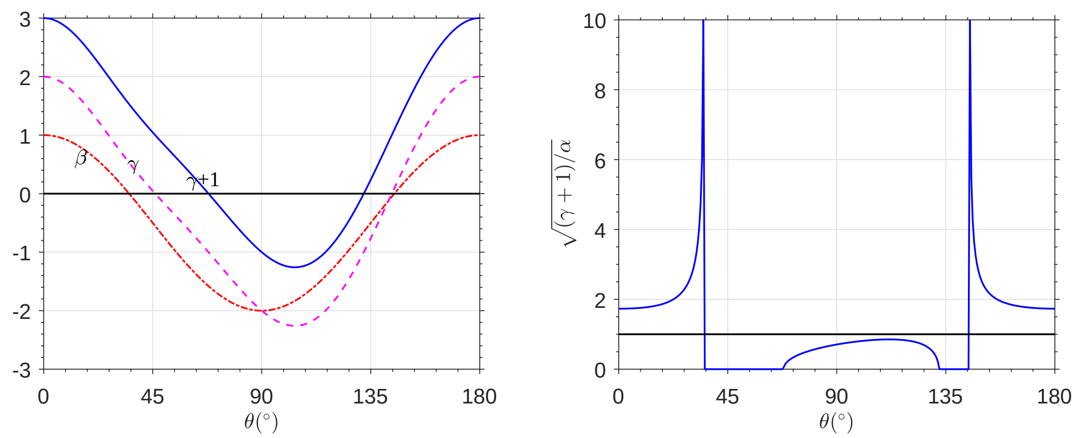


Figure 1: The variation of the coefficients in $\mathrm{CBFI}_{\mathrm{2d}}$ as a function of propagation direction $\theta$. Note that the crossing angle $\beta$ is equivalent to $2\theta$.

## 3 Examination of $\mathrm{CBFI}_{\mathrm{2d}}$ for random crossing waves

The purpose of the following tests is to demonstrate if the predictions of kurtosis by $\mathrm{CBFI}_{\mathrm{2d}}$ are valid. We conducted numerical simulations using a higher-order spectral method (Dommermuth & Yue, 1987; West *et al.*, 1987) of two identical crossing waves characterized each by JONSWAP frequency spectra (peak period $T_p = 1$ s, significant wave height $H_s = 0.06$ m, i.e. wave steepness is fixed $\varepsilon = 0.12$) and cosine-squared directional distribution (see Xiao *et al.*, 2013). The crossing angle between two wave systems is fixed $\beta = 40°$. To include a wider range of sea states, different enhancement factor ($\gamma = 2, 5$ and 8) and spreading bandwidth ($\Theta = 5°, 15°, 30°$ and $60°$) are considered here. For long-crested waves (small $\Theta$) with narrow-band frequency spectra (large $\gamma$), the kurtosis value is relatively larger and the rogue wave is more likely to be formed in such sea state . with the increases of directional spreading width and frequency width, the kurtosis value is reduced due to the suppression of four-wave resonant interactions (Onorato *et al.*, 2009). Numerical simulations were carried out in a two-dimensional domain of $32\lambda_p \times 32\lambda_p$ with spatial mesh of $1024 \times 1024$ nodes, where $\lambda_p$ is the wavelength corresponding to the peak period.

Besides, the $\mathrm{CBFI}_{\mathrm{2d}}$ is also examined using available experimental data. Toffoli *et al.* (2011) conducted a laboratory experiment of two crossing long-crested waves with varied crossing angle $\beta = 10°, 20°, 30°, 40°$ and fixed wave steepness $\varepsilon = 0.13$ to investigate the effect of crossing angle. Recently, to study the effect of directional spreading, Luxmoore *et al.* (2019) carried out a series of experiments on directional crossing waves.

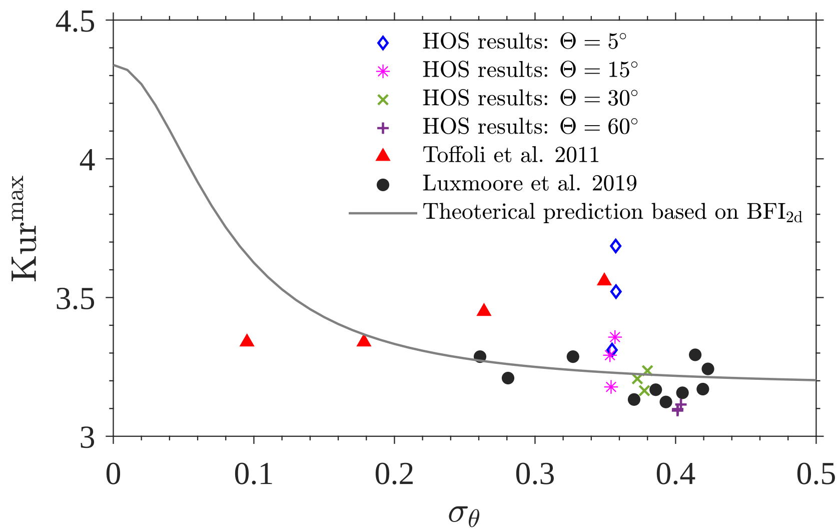


Figure 2: The maximum observed kurtosis in the total duration versus the directional spreading at the same time/position.

Figure 2 shows the maximum observed kurtosis in the total duration versus the directional spreading at the same time/position. Experimental results as well as present HOS results for the cases are presented. The theoretical prediction based on $\mathrm{BFI}_{\mathrm{2d}}$ (see Mori *et al.*, 2011) is plotted for comparison. Our results show that in the crossing seas with relatively broad-banded directional spreading ($\Theta = 15°, 30°$ and $60°$) the kurtosis value can be estimated quite well from directional spreading based on $\mathrm{BFI}_{\mathrm{2d}}$, which is in consistent with the finding in Luxmoore *et al.* (2019). However, in the nearly long-crested cases (e.g., the experiment by Toffoli *et al.*

(2011) and present simulation with $\Theta = 5°$) it is found that the kurtosis is underestimated by the theoretical prediction. This is attributed to that, the classical $\mathrm{BFI}_{2d}$ cannot capture the effect of crossing angle reasonably, which is more significant in nearly long-crested crossing waves.

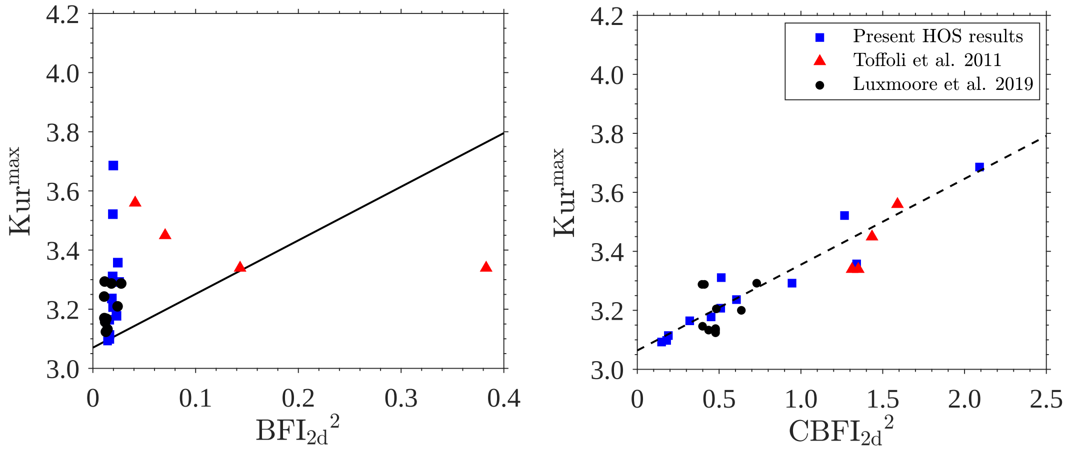


Figure 3: Dependence of kurtosis on $\mathrm{CBFI}_{2d}$ and $\mathrm{BFI}_{2d}$. Black solid line corresponds to the theoretical prediction based on $\mathrm{CBFI}_{2d}$ (1). Dashed line represents the linear regression results $\mathrm{Kur}=0.29\times\mathrm{CBFI}_{2d}{}^2+3.06$ ($R^2$=0.67).

Figure 3 illustrates the dependence of kurtosis on $\mathrm{CBFI}_{2d}$ and $\mathrm{BFI}_{2d}$ for crossing seas. Black solid line corresponds to the theoretical prediction. For all the cases including both experimental and our numerical results, with $\mathrm{CBFI}_{2d}$ the scatter in the data is greatly reduced, resulting in a clear and almost linear parameterization of kurtosis and $\mathrm{CBFI}_{2d}{}^2$ over a wide range of crossing sea states. Based on linear regression analysis, the semi-empirical formula is found to $\mathrm{Kur}=0.29\times\mathrm{CBFI}_{2d}{}^2+3.06$ with the related coefficient $R^2$=0.67. This result confirms that $\mathrm{CBFI}_{2d}$ gives a satisfactory indicator of third-order nonlinearity. To the authors' knowledge, it is the first time to develop a modified Benjamin-Feir index for two-component crossing seas and validate its relationship with the kurtosis value. More comprehensive analyses and discussion will be presented in the workshop.

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