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Third-order wave run-up in diffraction-radiation problems

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Introduction

Thanks to third-order approximation of gravity waves, it has been shown in the past that the interactions of two or several wave trains introduce a phase velocity change which occurs for each wave component due to the presence of the other components. This result established by Longuet-Higgins and Phillips [1] was used in the context of wave-structure interaction by Molin et al. [2]; assuming tertiary interactions between the incident field and the perturbated one by the structure (diffraction-radiation). Since then more research efforts have been made to better understand tertiary wave run-up in regular and irregular waves [3][4][5]. Some of these works have been presented and discussed on many occasions during the last previous workshops [2][6][4].

The main purpose of the present work is to introduce a numerical model accounting for third-order interactions in linear diffraction-radiation analysis. Where the methodology consists in coupling the existing parabolic model introduced by Molin et al [7] with a 3D BEM seakeeping code. Furthermore, the model is extended to irregular waves. The obtained numerical results show an excellent agreement with model tests in both regular and irregular waves, allowing for better estimation of wave elevation compared to linear theory.

Tertiary interactions between two plane waves

Here we recall the third-order theory of Longuet-Higgins and Phillips in deep water. We consider two regular waves of different frequencies ω_1 and ω_2 and relative heading β. First order potential $\phi^{(1)}$ and free-surface elevation $\eta^{(1)}$ are given by:

$$\eta^{(1)} = A_1 \cos(k_1 x - \omega_1 t) + A_2 \cos(k_2 x \cos\beta + k_2 y \sin\beta - \omega_2 t) \tag{1}$$

$$\phi^{(1)} = \frac{A_1 g}{\omega_1} e^{k_1 z} \sin(k_1 x - \omega_1 t) + \frac{A_2 g}{\omega_2} e^{k_2 z} \sin(k_2 x \cos\beta + k_2 y \sin\beta - \omega_2 t) \tag{2}$$

With A_1 and k_1 (respectively A_2 and k_2) being the amplitude and wave number of the ω_1-wave (respectively the ω_2-wave). Longuet-Higgins and Phillips have showed, after a lengthy perturbation analysis and retaining $(k_1 x - \omega_1 t)$ terms only, that the free surface condition satisfied by third-order potential $\phi^{(3)}$ can be written as [1]:

$$\phi^{(3)}_{tt} + g\phi^{(3)}_z = \left(A_1 A_2^2 \omega_1 \omega_2^2 k_1 \mathcal{F}(\omega_1,\omega_2,\beta) - A_1^3 \omega_1^3 k_1\right) \sin(k_1 x - \omega_1 t) \tag{3}$$

Where the right hand side includes two terms: the first one resulting from cross-interaction between the two waves and the second one resulting from self-interaction of the ω_1-wave. $\mathcal{F}$ is the so-called interaction function defined by Molin et al. [3] which depends on both wave frequencies ω_1 and ω_2, and the relative heading β:

$$\begin{aligned}\mathcal{F}(\omega_1,\omega_2,\beta) = &\frac{1}{2}(1+\cos^2\beta) - \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right)\cos\beta - \frac{k_2}{k_1} \\ &- \frac{\alpha^+ k^+}{2k_1}\left[\frac{k^+}{k_2} - \left(1 + \frac{\omega_1}{\omega_2}\right)^2\right] - \frac{\alpha^- k^-}{2k_1}\left[\frac{k^-}{k_2} - \left(1 - \frac{\omega_1}{\omega_2}\right)^2\right] \\ &- \frac{\omega_1}{\omega_2}\left[\frac{k_2}{k_1}\left(\alpha^+ - \alpha^-\right) + \cos\beta\ \left(\alpha^+ + \alpha^-\right)\right] - \frac{\omega_1}{\omega_2}\left(\frac{\alpha^+ k^+}{k_1} - \frac{\alpha^- k^-}{k_1}\right)\end{aligned} \tag{4}$$

With:

$$\begin{aligned} k^+ &= \sqrt{k_1 + k_2 + 2k_1 k_2 \cos\beta} \qquad & \alpha^+ &= \frac{\omega_2(\omega_1+\omega_2)}{gk^+ - (\omega_1+\omega_2)^2}(1-\cos\beta) \\ k^- &= \sqrt{k_1 + k_2 - 2k_1 k_2 \cos\beta} \qquad & \alpha^- &= \frac{\omega_2(\omega_2-\omega_1)}{gk^- - (\omega_1-\omega_2)^2}(1+\cos\beta)\end{aligned} \tag{5}$$

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Figure 1 shows $\mathcal{F}$ variation with the heading β for different frequency ratios $\alpha = \omega_1/\omega_2$. It is noticeable that tertiary interaction becomes particularly significant when $\omega_2 \to \omega_1$ and reach its a maximum at $\beta = \pi$. On the other hand, no interaction effects can be seen around $\beta \approx \pi/2$ (more precisely β=92.03°).

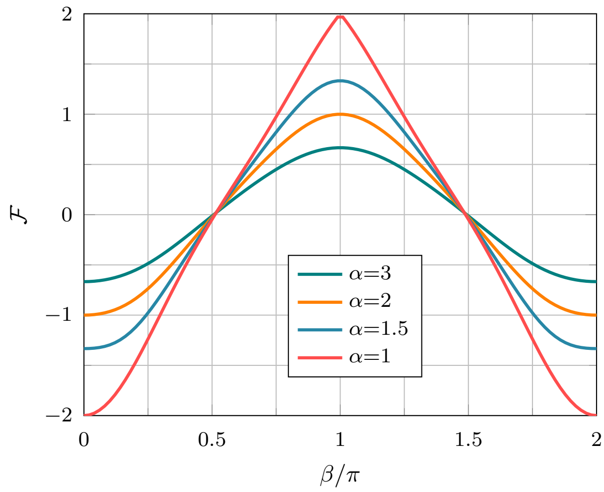

Figure 1: Interaction function $\mathcal{F}$ vs. β for different frequency ratios

Finally, the wave number change of the ω_1-wave induced by the ω_2-wave is then given by:

$$k_1^{(2)} = A_2^2 k_1^2 k_2 \mathcal{F}(\omega_1, \omega_2, \beta) \tag{6}$$

Accounting for tertiary interactions in diffraction-radiation problem

In linear potential flow theory, the total velocity potential ϕ_{tot} is decomposed into an incident part ϕ^I and a perturbated part ϕ^P including the diffraction potential ϕ^D and the 6 rigid motions radiation potentials ϕ^{R_j}:

$$\phi^{tot} = \phi^I + \phi^P = \phi^I + \phi^D - i\omega \sum_{j=1}^{6} \xi_j(\omega)\phi^{R_j} \tag{7}$$

Where ξ_j is the complex amplitude of the j$^{\text{th}}$ rigid motion. For a regular wave of frequency ω, linear incident potential is defined as:

$$\phi^I = -\frac{ig}{\omega} A^I e^{(kz+ikx-i\omega t)} \tag{8}$$

In order to take into account third order effects, Longuet Higgins and Phillips theory [1] is applied to the incident and perturbated fields. In this case, tertiary interactions are considered at the same frequency ω by introducing a modified incoming field $\tilde{\phi}^I$ as follow [7]:

$$\tilde{\phi}^I = -\frac{ig}{\omega} A(\epsilon^2 x, \epsilon y) e^{[k+\epsilon^2 k^{(2)}(\epsilon^2 x, \epsilon y)]z + i[k(1-\epsilon^2)x - \omega t]} \tag{9}$$

Where the amplitude A and the wave number $k^{(2)}$ are local in space, as a consequence of the spatial variation of the diffraction-radiation field. The Laplace condition gives the parabolic equation satisfied by A:

$$2ik\frac{\partial A}{\partial x} + \frac{\partial^2 A}{\partial y^2} + 2k^4 \left[{A^P}^2 \mathcal{F}(\omega, \omega, \beta) + {A^I}^2 - \|A\|^2 \right] A = 0 \tag{10}$$

With $A^P(x, y)$ and $\beta(x, y)$ are the amplitude and direction (both real) of the perturbated field, considered locally as a plane wave. $\|.\|$ is used to denote the complex modulus of A.

This parabolic equation is solved in 2D rectangular domain limited by tow fictive walls at $y = \pm b$, similar as a wave tank [7]. This implies no-flow condition at the walls and allows to expand A as a series of basis functions in y direction. Therefore, the integration is performed in x direction only starting from $x = -l$ where it is assumed that $A = A^I$ (no tertiary effects) and continued about 2-3 wavelengths past the structure.

To extend this theory to irregular waves, we consider a linear wave spectrum S with N frequency components. The linear incident field (velocity potential ϕ^I and free surface elevation η^I) is expressed as:

$$\eta^I = \sum_{n=1}^{N} \eta_n^I = \sum_{n=1}^{N} A_n^I e^{i(k_n x - \omega_n t + \theta_n)} \tag{11}$$

$$\phi^I = \sum_{n=1}^{N} \phi_n^I = \sum_{n=1}^{N} -\frac{ig}{\omega_n} A_n^I e^{k_n z + i(k_n x - \omega_n t + \theta_n)} \tag{12}$$

With θ_n the wave phase, ω_n the wave frequency and k_n the corresponding wave number. A_n^I is the wave amplitude given by $A_n^I = \sqrt{2S(\omega_n)\Delta\omega}$, $\Delta\omega$ being the frequency step.

In the same way as the regular wave case, Longuet-Higgins and Phillips theory [1] is applied to each frequency component ω_n. The modified incident potential of n$^{\text{th}}$ component is taken as follow:

$$\tilde{\phi}_n^I = -\frac{ig}{\omega_n} A_n(\epsilon_n^2 x, \epsilon_n y) e^{[k_n + \epsilon_n^2 k_n^{(2)}(\epsilon_n^2 x, \epsilon_n y)]z + i[k_n(1-\epsilon_n^2)x - \omega_n t + \theta_n]} \tag{13}$$

Here, $\epsilon_n = k_n A_n^I$ is the wave steepness of the n$^{\text{th}}$ frequency component and $k_n^{(2)}$ its wave number modification. The latter results from 3$^{\text{rd}}$ order interactions with the (N-1) other incident wave components and all the N perturbated wave components (diffraction+radiation). With this in mind, we can write:

$$k_n^{(2)} = k_n^2 \mathcal{I}_n = \sum_{m=1, m\neq n}^{N} k_m k_n^2 \|A_m\|^2 \mathcal{F}(\omega_n, \omega_m, 0) + \sum_{m=1}^{N} k_m k_n^2 {A_m^P}^2 \mathcal{F}(\omega_n, \omega_m, \beta_m) \tag{14}$$

Where A_m^P and β_m are respectively the amplitude and heading of the m$^{\text{th}}$ perturbated field component, idealized locally as a plane wave. Finally, Laplace condition yields to N parabolic equations to solve:

$$2ik_n \frac{\partial A_n}{\partial x} + \frac{\partial^2 A_n}{\partial y^2} + 2k_n^2 \epsilon_n^2 A_n + 2k_n k_n^{(2)} A_n = 0 \tag{15}$$

Taking $A_n = A_n^I(1 + a_n)$ [7], we obtain:

$$\frac{\partial a_n}{\partial x} - \frac{i}{2k_n} \frac{\partial^2 a_n}{\partial y^2} - ik_n(\epsilon_n^2 + k_n \mathcal{I}_n) a_n = ik_n(\epsilon_n^2 + k_n \mathcal{I}_n) \tag{16}$$

a_n is then expanded as a series of basis functions using the no-flow condition at the walls $y = \pm b$:

$$a_n = \sum_{l=0}^{\infty} \alpha_{nl}(x) \cos(\lambda_l(y + b)) = \sum_{l=0}^{\infty} \alpha_{nl}(x) \psi_l(y) \tag{17}$$

With $\lambda_l = l\pi/(2b)$. The projection over this basis functions gives the non-linear system of equations to solve:

$$\frac{\partial \alpha_{nl}}{\partial x} + \frac{i\lambda_l^2}{2k_n} \alpha_{nl} - \frac{ik_n}{(1 + \delta_{0l})b} \sum_{m=0}^{\infty} \left[\int_{-b}^{b} (\epsilon_n^2 + k_n \mathcal{I}_n) \psi_l(y) \psi_m(y) dy \right] \alpha_{nm} = \frac{ik_n}{(1 + \delta_{0l})b} \left[\int_{-b}^{b} (\epsilon_n^2 + k_n \mathcal{I}_n) \psi_l(y) dy \right] \tag{18}$$

Where δ_{ij} is the Kronecker symbol. It is important to note that $\mathcal{I}_n$ terms depend on the incident field amplitudes $\|A_m\|$ of all other frequency components. Therefore, parabolic equation (15) should be solved iteratively for all frequencies at each x value. To sum up, the iterative procedure can be described as follow: Starting from linear incident potential ϕ^I, the perturbated potential ϕ^P is obtained through diffraction-radiation computation. Then based on this ϕ^P, the parabolic equation is solved to calculate $\tilde{\phi}^I$ which is used, once again, as an input for diffraction-radiation analysis to get a new ϕ^P . This operation is repeated until convergence of $\tilde{\phi}^I$.

Preliminary results and discussions

Our numerical model is used to investigate wave run-up upon rectangular barge in regular/irregular waves [4]. The experiments have been performed in the Deepwater Wave Basin at Shanghai Jiao Tong University. The wave basin is 50 m long and 40 m wide. The water depth was set to 10 m. The barge model consists of two identical fixed boxes each 3.333 m long and 0.767 m wide. Gap resonance effects being ignored here, we can consider the model as one single box 1.601 m wide. The draught was 0.185 m.

Irregular waves were generated based on white noise spectrum with significant wave height of 41.7 mm. For numerical simulations, a flat spectrum has been used with frequency ranging from 0.3 Hz to 1.7 Hz. It must be pointed out that for irregular wave case, incident wave amplitudes depend on spectrum discretization. Therefore, convergence tests with respect to the number of frequency components N need to be performed. Figure 3 shows total wave elevation at the box front face center (indicated by a red mark in figure 2) with RAOs given for 5 frequencies using different discretizations. We can conclude that convergence can be achieved within N =17.

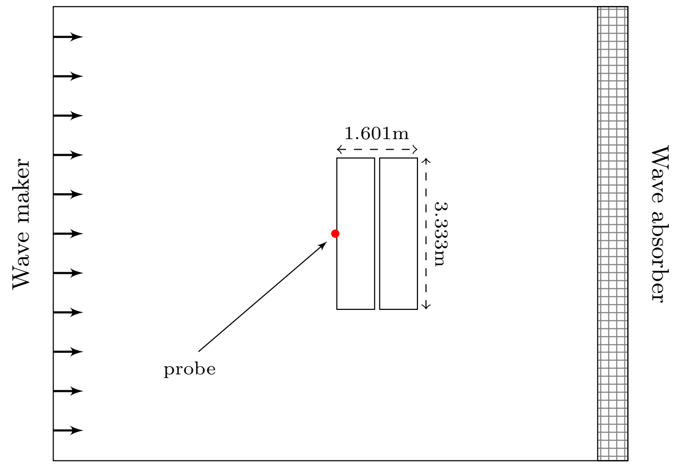

Figure 2: Experimental setup

Figure 4 compares total wave elevation at the same location; where the RAOs from experiments have been obtained using spectral analysis of measured signals over individual windows [4]. For more clarity, mean and range boundaries of the these RAOs have been plotted instead of representing all of them. A good agreement with experiments has been found. Again, as demonstrated by Molin for regular waves, accounting for tertiary wave effects in wave-body interactions provides better estimation of wave elevation compared to linear theory, even in irregular waves. More detailed results will be presented and discussed at the workshop.

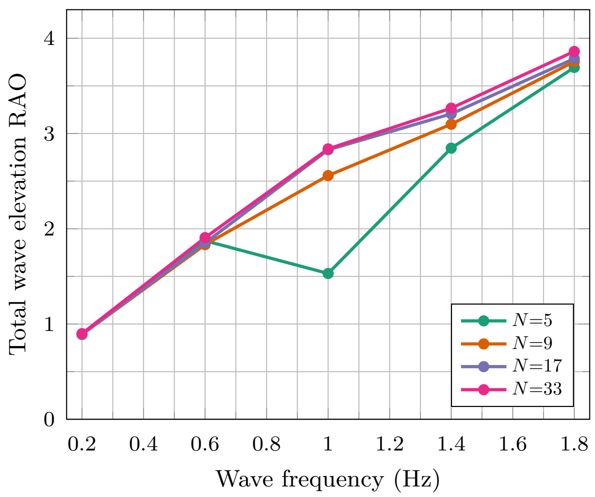

Figure 3: Sensitivity to spectrum discretization, total wave elevation RAO at the box front face center

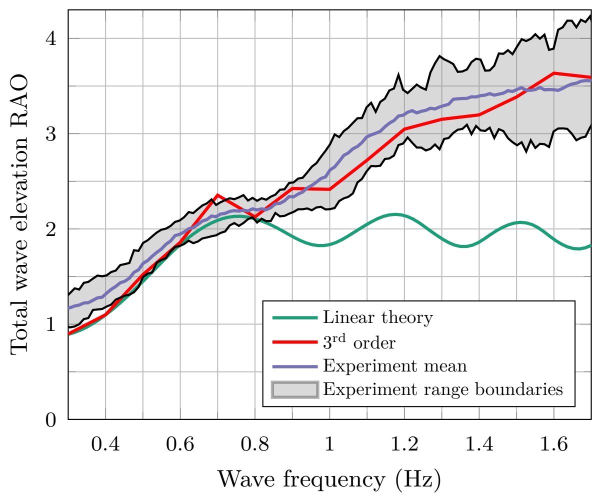

Figure 4: Comparison of total wave elevation RAO at the box front face center

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