Miloh's image singularities for oblate spheroids - A method developed for the water wave diffraction and radiation problems

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Introduction

The main objective of this study is to develop a semi-analytical formulation for the hydrodynamic diffraction and radiation problems of a fully submerged spheroid. The term "spheroid" refers here to the oblate geometry of arbitrary eccentricity and to the axisymmetric configuration, i.e. the spheroid's axis of symmetry is oriented perpendicular to the undisturbed free surface. The spheroid is assumed to be immersed below the free surface and subjected to harmonic incident waves, while the examined water depth is assumed to be infinite. The proposed approach is based on using the method of the image singularity system which, for the case of a spheroid, contains multipole expansions in terms of external spheroidal harmonics, allocated lengthwise the major axis of the spheroid between its two foci.

The method of the image singularities, developed initially for a prolate spheroid, was suggested for the first time without a proof by Havelock (1952). The proof was later obtained rigorously by Miloh (1974). Accordingly, Chatjigeorgiou and Miloh (2015, 2014a and 2014b) based their formulation on the method of the image system singularity, when studying the radiation and diffraction hydrodynamic problem of prolate spheroids with non-axisymmetric placement (i.e., spheroid axis parallel to the free surface). Only recently Chatjigeorgiou (2018) reported the corresponding proof of the image system singularities for oblate spheroidal geometries. Nevertheless, as of today, the associated theorem has not been employed numerically, a glitch that is remediated herein.

The novelty of the effort conducted in the present study lies in the development of a semi-analytical approach based on the method of the image singularity system, which is applied to axisymmetric oblate spheroidal configurations. The recommended semi-analytical approach outweighs the conventional numerical methods in terms of computational effort, while it offers a precise solution to the problem. The proposed methodology is applicable to spheroidal-like geometries encountered in practice, such as wave energy converters.

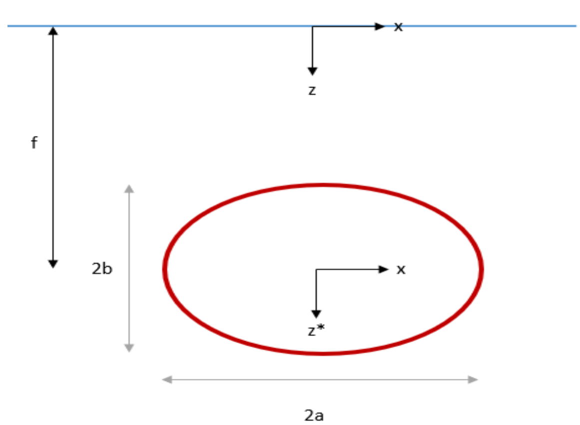

Fig. 1 Problem's set up.

The boundary value problem

The oblate spheroid is assumed to be immersed at a distance f below the undisturbed free surface (Fig. 1). A Cartesian (x, y, z) coordinate system is placed on the free surface and a second (x, y, z^*) Cartesian coordinate system is placed at the center of the body. In both cases the vertical axes are pointing downwards, so that $z = z^* + f$. The transformation from oblate spheroidal to Cartesian coordinates is $x = c\cosh u \sin\theta\cos\psi$, $y = c\cosh u\sin\theta\cos\psi$ and $z^* = c\sinh u\cos\theta$, where $0 \le u \le \infty$, $0 \le \theta \le \pi$ and $0 \le \psi \le 2\pi$. Defining $\xi = \sinh u$ and $\mu = \cos\theta$, one gets $x = c\sqrt{1+\xi^2}\sqrt{1-\mu^2}\cos\psi$, $y = c\sqrt{1+\xi^2}\,\sqrt{1-\mu^2}\sin\psi$ and $z^* = c\xi\mu$, while c denotes the semi-focal distance and is equal to $c = \sqrt{a^2 - b^2}$, where a and b are the semi-major and the semi-minor axis of the spheroid. For simplicity, we assume that $c = 1$ which is also used in the sequel as a reference length scale.

The fluid is assumed inviscid, incompressible and the flow field irrotational. These assumptions allow us to introduce a linear velocity potential of the form $\Phi(x, y, z, t) = Re(\phi e^{-i\omega t})$, where ϕ denotes the time independent complex potential and ω is the circular frequency of motion. The velocity potential ϕ must satisfy the subsequent (linearized) boundary value problem

$$\nabla^2\phi = 0, \text{ in } \Omega, \tag{1}$$

$$K\phi + \frac{\partial\phi}{\partial z} = 0, \qquad \text{on } S_F, \qquad z = 0, \tag{2}$$

$$\phi \to 0, \qquad z \to \infty, \tag{3}$$

where $K = \omega^2/g$, g is the gravitational acceleration, Ω is the semi-infinite liquid domain and S_F is the undisturbed free surface. Eq. (1) is the Laplace equation, Eq. (2) is the common linear free surface (Robin type) boundary condition and Eq. (3) ensures that the potential is zero to infinity away from the free surface. The body boundary conditions for the diffraction and radiation problems, respectively take the following form:

$$\frac{\partial(\phi_D + \phi_I)}{\partial n} = 0, \tag{4}$$

where ϕ_D, ϕ_I are the diffraction and the incident velocity potentials. The standard conditions for the radiation potentials will read

$$\frac{\partial\phi_j}{\partial n} = n_j, \qquad \frac{\partial\phi_{j+3}}{\partial n} = (\vec{r}\times\vec{n})_j, \tag{5}$$

where ϕ_j is the radiation potential for each mode of motion, i.e. three translational and three rotational motions, n_j is the unit vector in the respective direction j and $\vec{r}$ is the position vector.

In the present work we examine the axisymmetric case of placement of the oblate spheroid. Hence, due to symmetry only the surge, heave and pitch motions are taking into consideration. The outward surface normals for the examined motions (surge, heave, pitch) are

$$(n_1, n_3, n_5) = \left(\frac{\xi_0(1-\mu^2)^{1/2}\cos\psi}{(\xi_0^2+\mu^2)^{1/2}}, \frac{\mu(\xi_0^2+1)^{1/2}}{(\xi_0^2+\mu^2)^{1/2}}, \frac{c\mu(1-\mu^2)^{1/2}\cos\psi}{(\xi_0^2+\mu^2)^{1/2}}\right). \tag{6}$$

The boundary value problem is completed by employing the radiation condition for outgoing waves at infinity.

Solution using the image singularity system in oblate geometry

The velocity potential that actually governs the diffraction and the radiation problems can be written as

$$\phi = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m\, G_n^m \tag{7}$$

where A_n^m are unknown expansion coefficients and G_n^m denotes the auxiliary Green's functions (expansions based on multipoles). The values of A_n^m will be obtained using the body conditions (4) for the diffraction and (5) for the radiation problems. The multipoles G_n^m result from the underlying particular Green's function of the concerned hydrodynamic problem. To this end, the Green's function should be processed using the image singularity system for oblate spheroids (Chatjigeorgiou, 2018). Following Wehausen and Laitone (1960), we select to express the corresponding Green's function in the following form:

$$G(x-x',y-y',z) = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-f)^2}} - \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} \frac{K+k}{K-k} e^{-k(z+f)} e^{ik[(x-x')\cos a+(y-y')\sin a]} da dk. \quad (8)$$

Clearly, Eq. (8) satisfies the Laplace equation (1), the free surface condition (2) and the far-field condition (3). The wave radiation condition is next satisfied by interpreting the infinite improper integral in Eq. (8) as a Cauchy Principal Value integral, by defining (see Wehausen and Laitone, 1960)

$$\int_0^\infty \frac{F(k)}{\sigma-k} dk = PV\int_0^\infty \frac{F(k)}{\sigma-k} dk - i\pi F(\sigma). \quad (9)$$

The sought multipoles G_n^m in Eq. (7) can then be obtained by processing Eq. (8) using Miloh's theorem for image singularities in oblate geometry. The proof of this theorem can be found in the book of Chatjigeorgiou (2018). The associated formula, that is connected with the external oblate spheroidal harmonics, reads

$$P_n^m(\mu)Q_n^m(i\xi_0)\cos m\psi = \frac{(-1)^m}{2\pi P_n^m(i0)}\frac{(n+m)!}{(n-m)!}\int_0^{2\pi}\int_{-1}^{1} \frac{P_n^m(\mu')\cos m\psi'}{\sqrt{(x-x')^2+(y-y')^2+z^2}} d\mu' d\psi'. \quad (10)$$

The above formula holds for $n-m$ even, which corresponds to symmetry with respect to the z-axis. Eq. (10) allows the transformation of the singular term in Eq. (8) into the concerned harmonics. Note that P_n^m and Q_n^m denote the nth degree and mth order of the associate Legendre functions of the first and the second kind respectively. The Green's function must be manipulated further using the following transformation of the exponential term into oblate spheroidal harmonics:

$$e^{kz^*+ik(x\cos a+y\sin a)} = \sum_{s=0}^{\infty}\sum_{t=0}^{s}(A_s^t\cos t\psi + B_s^t \sin t\psi)\, P_s^t(\mu)P_s^t(i\xi), \quad (11)$$

where

$$\begin{pmatrix} A_s^t \\ B_s^t \end{pmatrix} = (-1)^s i^{s-t}\varepsilon_t \frac{(s-t)!}{(s+t)!}\sqrt{\frac{\pi}{2k}} J_{s+1/2}(k)\begin{pmatrix}\cos ta \\ \sin ta\end{pmatrix}, \quad (12)$$

and J_ν denotes the Bessel function of the first kind. Doing so, the Green's function (8) is fully converted to be expressed relative to the underlying harmonics. The auxiliary G_n^m are eventually given by

$$G_n^m = P_n^m(\mu)Q_n^m(i\xi)\cos m\psi + \sum_{s=0}^{\infty}\sum_{t=0}^{s} C_{ns}^{mt}\, P_n^m(\mu)P_n^m(i\xi)\cos t\psi, \quad (13)$$

where

$$C_{ns}^{mt} = -\frac{1}{4}(-1)^{n+s} i^{n+s-m-t} \frac{2\pi}{\varepsilon_m} \varepsilon_t \frac{(n+m)!\,(s-t)!}{(n-m)!\,(s+t)!}(2s+1)\delta_{tm}$$
$$\times \left(PV \int_0^\infty \frac{K+k}{K-\kappa} e^{-2kf} \frac{1}{k} J_{n+1/2}(k) J_{s+1/2}(k)\, dk - 2\pi i e^{-Kf} J_{n+1/2}(K) J_{s+1/2}(K) \right) \tag{14}$$

Note that $\varepsilon_0 = 1$ and $\varepsilon_n = 2, n = 1,2, \ldots$ denotes the Neumann symbol and δ_{tm} is the Kroneker's delta function. The final expression for the velocity potentials (diffraction or radiation) is given by

$$\phi = \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_n^m P_n^m(\mu) Q_n^m(i\xi) \cos m\psi + \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_n^m \sum_{s=0}^{\infty} \sum_{t=0}^{s} C_{ns}^{mt} P_s^t(\mu) P_s^t(i\xi) \cos t\psi \tag{15}$$

which allows us to derive the coefficients A_n^m using Eqs. (4) and (5). The forces and the hydrodynamic parameters can, then, be readily obtained by a direct pressure integration on the wetted surface of the solid. Some relevant results for the exiting hydrodynamic loading, obtained using the present methodology are shown in Fig. 2. It is important to note that they show remarkable coincidence with the WAMIT computation code. The relatively fast convergence (using only five modes) of the results manifests the efficacy and accuracy of the newly proposed image singularity method for oblate shapes.

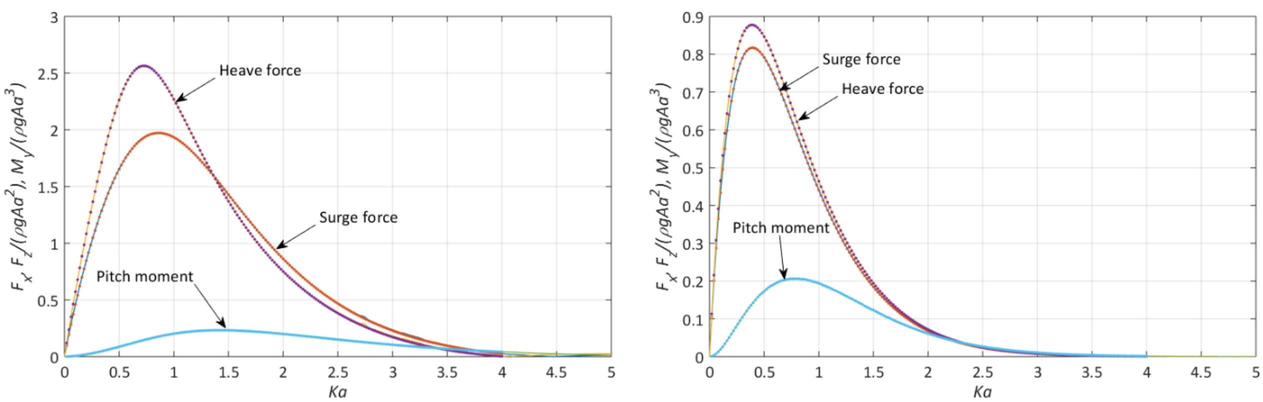

Fig. 2 Magnitudes of the hydrodynamic loading acting on an oblate spheroid $b/a = 0.8$, $f/a = 1$ (left) and $b/a = 0.9$, $f/a = 2.5$ (right). The symbols denote the results of WAMIT.

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