An extension to the linear shallow water equation

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1 Introduction

The linearised Shallow Water Equation (SWE) approximates the propagation of surface gravity waves over variable bathymetry $z = -h(x, y)$ in the long wavelength limit, $\lambda \gg h$, and is commonly expressed (e.g. Stoker (1957)) in the form

$$g\nabla \cdot (h\nabla\zeta) = \zeta_{tt} \tag{1}$$

where g is acceleration due to gravity, $\nabla = (\partial_x, \partial_y)$ and $\zeta(x, y, t)$ is the free surface elevation assumed to be small in the sense that $|\nabla\zeta| \sim \zeta/\lambda \ll (h/\lambda)^3$. This latter assumption justifies the linearisation of the governing equations in what follows.

When time-harmonic motion is considered and $\zeta(x, y, t) = \Re\{\eta(x, y)e^{-i\omega t}\}$, (1) is transformed to

$$\nabla \cdot (h\nabla\eta) + K\eta = 0 \tag{2}$$

where $K = \omega^2/g$. The local wavenumber $k(x, y) = 2\pi/\lambda$, determined by the local depth $h(x, y)$ as though the bed were flat, satisfies $k^2h = K$ and this corresponds to the long wavelength $(kh \to 0)$ limit of the unapproximated water wave dispersion relation $k \tanh kh = K$. Under the SWE waves are non-dispersive.

The SWE is practically limited to modelling of very long waves, for example tidal simulations or tsunami wave propagation. In spite of this the SWE has received renewed recent attention on account of its structural similarity to 2nd order partial differential equations describing waves in two-dimensional acoustics and TM- or TE-polarised electromagnetics and this analogue has seen it used as a model for producing exotic effects in water waves such as invisibility cloaking, negative refraction, wave-shifting and other wave control mechanisms (see, for e.g., the review in Porter (2018).) Often studies of these topics are accompanied by experiments which are of questionable quality − understandable not least because the conditions of shallow water theory are not easily met. This work is partly aimed at addressing this.

In the classical derivation of (1) the vertical acceleration, W_t, is neglected from the momentum equations since it can be shown to contribute at $O(\mu^2)$ where $\mu = H/L \ll 1$ and H, L are characteristic depth and horizontal lengthscales (see Stoker (1957).) However the contribution from W itself *is* required to satisfy the mass conservation equation. The retention of W_t in the momentum equations is not new and central to the formulation of Boussinesq-type equations for water waves (see Peregrine (1967)) where weakly-nonlinear effects are also included in the governing equations by assuming an Ursell number of $O(1)$.

As far as the author is aware, the vertical acceleration has not been retained in the momentum equations in a linearised setting (Ursell number $\ll 1$) and this is the subject of the current work. There are a number of good reasons for doing this which are described in the summary at the end of this paper.

2 Formulation

Cartesian coordinates are used in which the bed is given by $z = -h(x, y)$ and the free surface by $z = \zeta(x, y, t)$. Following Stoker (1957) the horizontal components of the flow are assumed to be independent of the depth variable, consistent with the assumption of a long wavelength compared to the depth, and a flow velocity of

$$\mathbf{u} \approx (\mathbf{U}, W) \tag{3}$$

is assumed where $\mathbf{U} = (U(x,y,t), V(x,y,t))$ and

$$W(x,y,z,t) = (z/h+1)\zeta_t + (z/h)\nabla h \cdot \mathbf{U} \tag{4}$$

is linear in z allowing the kinematic conditions on the surface and the bed, namely

$$W = \zeta_t \quad \text{on } z = 0 \qquad \text{and} \qquad W + \nabla h \cdot \mathbf{U} = 0 \quad \text{on } z = -h(x,y), \tag{5}$$

to be satisfied exactly. Integration of the continuity equation $W_z + \nabla \cdot \mathbf{U} = 0$ over the depth gives the familiar shallow water condition

$$\zeta_t = -\nabla \cdot (h\mathbf{U}) \tag{6}$$

since this result only requires (5) and does not depend on the specification of W in (4). Taking ρ as the fluid density and with $p(x,y,z,t)$ representing the fluid pressure, the vertical component of the linearised momentum equation, $\rho W_t = -p_z - \rho g$, integrates to

$$p(x,z,t) = p_a + \rho g(\zeta - z) - \rho\left(\frac{z^2}{2h} + z\right)\zeta_{tt} - \rho\frac{z^2}{2h}\nabla h \cdot \mathbf{U}_t \tag{7}$$

where p_a is atmospheric pressure. Using (7) in the horizontal linearised momentum equations, $\rho\mathbf{U}_t = -\nabla p$ gives, after depth averaging,

$$h\mathbf{U}_t = -gh\nabla\zeta - \tfrac{1}{3}h^2\nabla\zeta_{tt} - \tfrac{1}{6}h\{\zeta_{tt} + (\nabla h \cdot \mathbf{U}_t)\}\nabla h + \tfrac{1}{6}h^2\nabla(\nabla h \cdot \mathbf{U}_t) \tag{8}$$

and subsequent manipulations can be made to result in

$$h^{-1}\{\mathsf{I} + \tfrac{1}{3}\mathsf{h}'^2 - \tfrac{1}{6}h\mathsf{h}'' + \tfrac{1}{6}h\mathsf{D}\}\mathbf{Q}_{tt} = \nabla(g\nabla \cdot \mathbf{Q} + \tfrac{1}{3}h\nabla \cdot \mathbf{Q}_{tt}) \tag{9}$$

where $\mathbf{Q} = h\mathbf{U}$, $\zeta_t = -\nabla \cdot \mathbf{Q}$. In (9), I is the 2×2 Identity matrix, h'' represents the the Hessian of h and

$$\mathsf{h}'^2 \equiv (\nabla h)(\nabla h)^T = \begin{pmatrix} h_x^2 & h_x h_y \\ h_x h_y & h_y^2 \end{pmatrix} \quad \text{and} \quad \mathsf{D} = \begin{pmatrix} 0 & \mathbf{t} \cdot \nabla \\ -\mathbf{t} \cdot \nabla & 0 \end{pmatrix} \tag{10}$$

is an antisymmetric differential operator in which $\mathbf{t} = (-h_y, h_x)$ is directed along level curves of $h(x,y)$.

Thus, the extended SWE in the time domain and in three dimensions is represented by the vector equation (9) for $\mathbf{Q}$. In a two-dimensional setting in which $h = h(x)$ and there is no y-dependence elsewhere, $\mathbf{Q}(x,y,t) = Q(x,t)\hat{\mathbf{x}}$ and (9) reduces to the scalar equation

$$Q_{tt} = \hat{h}(x)(gQ_x + \tfrac{1}{3}h(x)Q_{ttx})_x \tag{11}$$

where $\hat{h}(x) = h(x)/(1 + \frac{1}{3}h'^2(x) - \frac{1}{6}h(x)h''(x))$ whilst $\zeta_t = -Q_x$ can be used to express this as

$$\zeta_{tt} = \left(\hat{h}(x)(g\zeta + \tfrac{1}{3}h(x)\zeta_{tt})_x\right)_x \tag{12}$$

which is close, but not identical, to the form expressed in (1).

More significant progress can be made once time harmonicity is assumed. Thus, returning to (9) and writing $\mathbf{Q}(x,y,t) = \Re\{\mathbf{q}(x,y)\mathrm{e}^{-\mathrm{i}\omega t}\}$ we have

$$\nabla((1 - \tfrac{1}{3}Kh)\nabla \cdot \mathbf{q}) + Kh^{-1}\{\mathsf{I} + \tfrac{1}{3}\mathsf{h}'^2 - \tfrac{1}{6}h\mathsf{h}'' + \tfrac{1}{6}h\mathsf{D}\}\mathbf{q} = 0. \tag{13}$$

When rescaled using

$$\mathbf{q}(x,y) = \boldsymbol{\varphi}(x,y)/\sqrt{1 - \tfrac{1}{3}Kh} \tag{14}$$

(13) gives, after considerable but routine algebra, the vector equation

$$\nabla(\nabla \cdot \boldsymbol{\varphi}) + (\hat{K}/h)\{\mathsf{I} + \tfrac{1}{3}v(h)\mathsf{h}'^2\}\boldsymbol{\varphi} = 0 \tag{15}$$

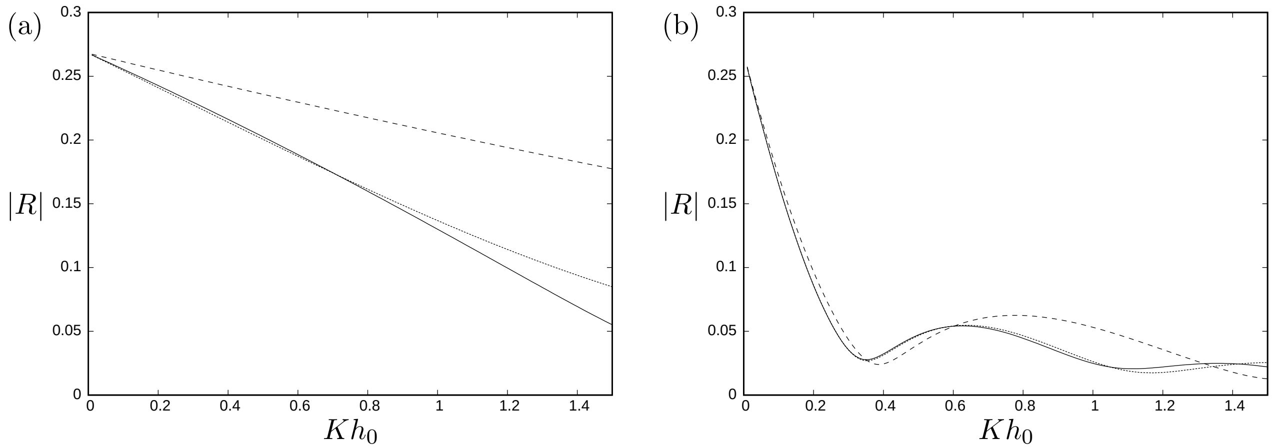

Figure 1: Modulus of reflection coefficient against Kh_0 for a linear ramp with $h_L/h_0 = \frac{1}{3}$ and $h_0/L = 1, \frac{1}{4}$ in (a),(b): full linear theory (dotted), standard SWE (dashed), extended SWE (solid).

where $v(h) = 1 + \frac{1}{12}\hat{K}h$ and $\hat{K} = K/(1 - \frac{1}{3}Kh)$. Finally, writing $\Omega(x, y) = \nabla \cdot \boldsymbol{\varphi}$ allows (15) to be converted into

$$\nabla \cdot (\hat{\hat{\mathsf{h}}}\nabla\Omega) + K\Omega = 0 \tag{16}$$

and this is the scalar version of the extended SWE in the frequency domain. It has the structure of (2) but with the scalar depth $h(x, y)$ replaced by the 2×2 matrix/tensor

$$\hat{\hat{\mathsf{h}}}(x, y) = h(x, y)(1 - \tfrac{1}{3}Kh)(\mathsf{I} + \tfrac{1}{3}v(h)\mathsf{h}'^2)^{-1} = h(x, y)(1 - \tfrac{1}{3}Kh)\left(\mathsf{I} - \frac{v(h)}{(3 + v(h)|\nabla h|^2)}\mathsf{h}'^2\right). \tag{17}$$

The surface elevation can be reconstructed from Ω using

$$\eta(x, y) = \frac{(-\mathrm{i}/\omega)}{\sqrt{1 - \frac{1}{3}Kh}}\left(\Omega - \tfrac{1}{6}h\nabla h \cdot \left((\mathsf{I} + \tfrac{1}{3}v(h)\mathsf{h}'^2)^{-1}\nabla\Omega\right)\right). \tag{18}$$

We note that the transformation of (13) to (15) via the scaling (14) removes the dependence on second derivatives of $h(x, y)$ from the governing equations and this has an additional advantage that both $\boldsymbol{\varphi}(x, y)$ and $\Omega(x, y)$ are continuous even when ∇h is not. It is also evident from (18) that discontinuities in ∇h imply discontinuities in the function $\eta(x, y)$ representing the surface elevation. This is simply a consequence of the particular approximation chosen and we remark that the standard SWE suffers from discontinuities in $\nabla\eta$ at such points.

3 Examples

We present results for two-dimensional scattering and two test cases. The first is a linear slope in the bed connecting the constant depth h_0 in $x < 0$ to the constant depth h_L for $x > L$. This is known as the Booij problem (after Booij (1983)) and we are interested in the variation of the reflection coefficient $|R|$ against frequency for waves incident from $x = -\infty$. Fig. 1 shows results for a shoaling parameter $h_L/h_0 = \frac{1}{3}$ and two different values of bed steepness over the shoal, $h_0/L = 1, \frac{1}{4}$. Results from the original SWE (2) and the extended SWE (16) are compared with accurate numerical results based on full linear theory computed using the method of Porter & Porter (2000). The extended SWE shows a significant improvement especially for steeper beds and weakly-dispersive nature of the solutions of (16) is also clearly evident.

A similar set of results are shown in Fig. 2 again for the bed profile considered by Roseau (1976) shoaling asymptotically from depth h_0 to h_L at either infinity. Roseau's profile, parametrised by β which encodes the steepness of the bed, results in an explicit formula for the reflection coefficient, namely

$$|R| = \left|\frac{\sinh[(k_0h_0 - k_Lh_L)/\beta]}{\sinh[(k_0h_0 + k_Lh_L)/\beta]}\right| \tag{19}$$

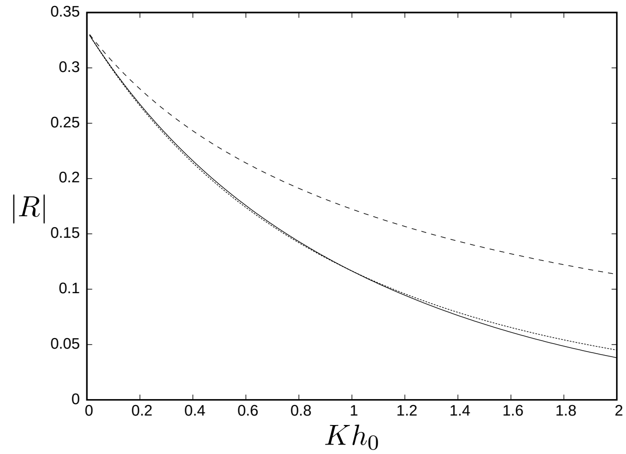

Figure 2: Modulus of reflection coefficient against Kh_0 for the Roseau problem with $h_L/h_0 = 0.25$, $\beta = 0.5$: full linear theory (dotted), standard SWE (dashed), extended SWE (solid).

where $k_0 \tanh k_0 h_0 = k_L \tanh k_L h_L = K$. Comparisons between (19) and the two versions of the SWE for $\beta = 0.5$ (a maximum bed gradient of 0.75) and $h_L/h_0 = \frac{1}{4}$ confirm the comments previously made on the superiority of the extended SWE over the standard version.

4 Summary

The key points of this work are: (i) the retention of higher-order terms and the exact satisfaction of the bed condition in the governing equations leads to an extension of the SWE which shows significant improvements in the accuracy and range of parameters of frequencies over which it can be applied; (ii) in the frequency-domain, the extended SWE can be readily implemented in place of the original SWE since it retains its structure with $\hat{\hat{\mathsf{h}}}$ defined by (18) replacing h; (iii) the extended SWE demonstrates anisotropy of wave scattering over variable beds (i.e. the wave speed is dependent, in general, on the wave heading) with potentially important consequences for the design of water-wave metamaterials; (iv) the extended SWE connects to the long-wavelength limit of the so-called Complementary Mild-Slope Equation of Kim & Bai (2004) and Toledo & Agnon (2010), also a depth-averaged model of wave scattering in which the bed condition is satisfied exactly but derived from a variational principle.

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