Steady solution of internal solitary waves in linear shear current

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HIGHLIGHTS:

- The steady solution of internal solitary waves in linear shear current is obtained by the two-layer GN model.
- The wave profile of internal solitary waves in linear shear current is studied. The velocity field and wave speed are also studied.

1 INTRODUCTION

Internal waves are of interest to engineers since they can induce large forces on moorings of offshore platforms and pipelines, influence propagation of acoustic signals, and so on. Many works have been done on internal solitary waves in a two-layer fluid system. Miyata (1985,1988) developed the internal-wave theory with a rigid lid for one-dimensional internal waves in a two-layer fluid system (see Choi and Camassa (1999)). Therefore, this strongly nonlinear model, called the MCC (Miyata, Choi, Camassa) model, is widely used since no assumption is made. Grue et al. (1999) used both experimental and theoretical methods to investigate the properties of solitary waves propagating in a two-layer fluid.

These above studies did not consider the effect of background shear current, which always exists in the real ocean. Research on internal solitary wave in shear current is rare. Choi (2006) firstly studied the steady solution of the internal solitary waves in linear shear flow, including the wave speed and wave profile. Results showed that for the case of negative vorticity, a solitary wave of depression travelling in the positive x direction is found to be wider and slower (compared with the irrotational case), while the solitary wave of depression is found to be narrower and faster when traveling in the negative x direction. However, the velocity field is not considered by Choi (2006). Also, no higher-order solutions or other results were used to test the accuracy of the results obtained by Choi (2006) .

The Green-Naghdi (GN) models only introduce an assumption on the velocity variation in the vertical direction across the fluid sheet. Consequently, based on different polynomial orders used for the description of velocity in the vertical direction, the GN theory can be classified into different levels, such as GN-1, GN-2, GN-3, and so forth. It was shown by Zhao et al. (2016) that high-level GN equations can provide a good prediction for steady solution of internal solitary waves, including the wave speed, wave profile and velocity distribution. Since there is no limitation on irrotationality, the GN equations can be used to solve the problem of the interaction between waves and linear shear current.

Here we study the steady solution of internal solitary waves in linear shear current, including various quantities of interest such as the wave profile, velocity field and wave speed. The governing equations are described in Section 2, algorithm is introduced briefly in Section 3, calculation results are presented in Section 4, and the conclusions we reach are in Section 5.

2 TWO-LAYER GN MODEL

Fig.1 shows the sketch of this physical problem. The GN equations are used in each of the two layers. In the GN model for one layer, the horizontal velocity is expressed approximately by $u(x,z,t)=\sum_{n=0}^{K-1}u_n(x,t)z^n$. Take K=3 as example, we have

$$u\left(x,z,t\right)=u_0\left(x,t\right)+u_1\left(x,t\right)z+u_2\left(x,t\right)z^2 \tag{1}$$

The GN-K^U–K^L equations (U stands for the upper fluid and L stands for the lower fluid) are as follows:

$$\frac{\partial\beta}{\partial t}=\sum_{n=0}^{K^U}\beta^n\left(w_n^U-\frac{\partial\beta}{\partial x}u_n^U\right) \tag{2}$$

$$\frac{\partial\beta}{\partial t}=\sum_{n=0}^{K^L}\beta^n\left(w_n^L-\frac{\partial\beta}{\partial x}u_n^L\right) \tag{3}$$

$$M_1^L+\frac{\rho^U}{\rho^L}(\beta-\alpha)\frac{\partial}{\partial x}\left(G_0^U+gS_0^U\right)-\frac{\rho^U}{\rho^L}\frac{\beta-\alpha}{\gamma-\beta}M_1^U=0 \tag{4}$$

$$\frac{M_n^U}{\gamma^n-\beta^n}-\frac{M_{n-1}^U}{\gamma^{n-1}-\beta^{n-1}}=0 \qquad n=2,3,\ldots,K^U \tag{5}$$

$$\frac{M_n^L}{\beta^n-\alpha^n}-\frac{M_{n-1}^L}{\beta^{n-1}-\alpha^{n-1}}=0 \qquad n=2,3,\ldots,K^L \tag{6}$$

For details on the GN equations, readers are referred to Zhao et al. (2016).

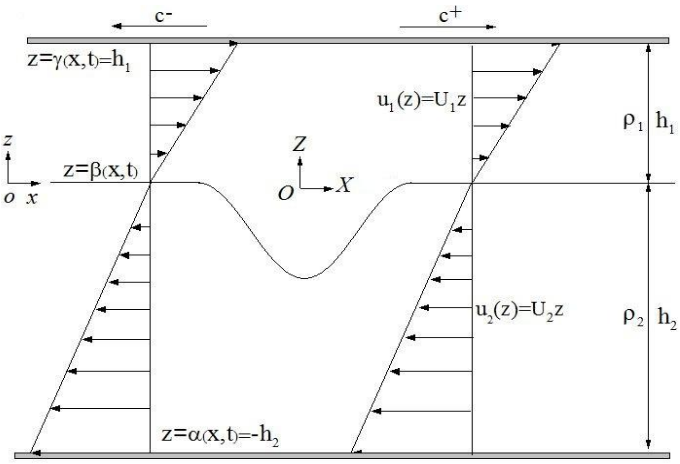


Fig.1 Sketch of the internal solitary wave in linear shear current

3 ALGORITHM

We use wave coordinates *XOZ* to obtain the steady solution of an internal solitary wave in shear current, where $X=x-ct$ (c is the wave speed). The *OX* axis located on the interface of the two fluids. The profile of linear shear current for the upper fluid and lower fluid is expressed by $u_1(z)=U_1 z$ and $u_2(z)=U_2 z$ respectively, where U_1 and U_2 are the current strength. When $X=0$ (wave crest), we have

$$\beta_{,X}=0,\ \ u_{n,X}^U=0\ (n=0,1,\ldots,K^U\text{-}1),\ \ u_{n,X}^L=0\ (n=0,1,\ldots,K^L\text{-}1). \tag{7}$$

where the comma means the derivative of β, u_n^U and u_n^L. When $X\to\infty$, we have

$$\beta=0,\ \ u_0^U=0,\ \ u_1^U=U_1,\ \ u_n^U=0\ \ (n=2,3,\ldots,K^U\text{-}1),\ \ u_0^L=0,\ \ u_1^L=U_2,\ \ u_n^L=0\ \ (n=2,3,\ldots,K^L\text{-}1). \tag{8}$$

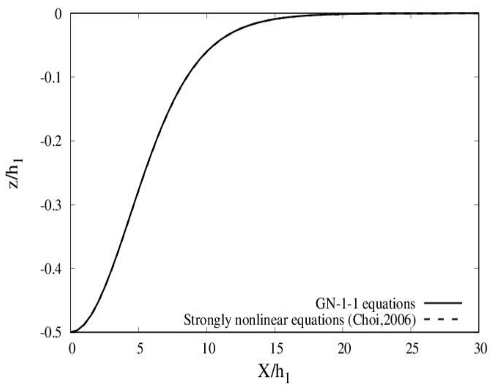
The steady solutions are obtained numerically by use of the Newton-Raphson method.

4 CALCULATION RESULTS

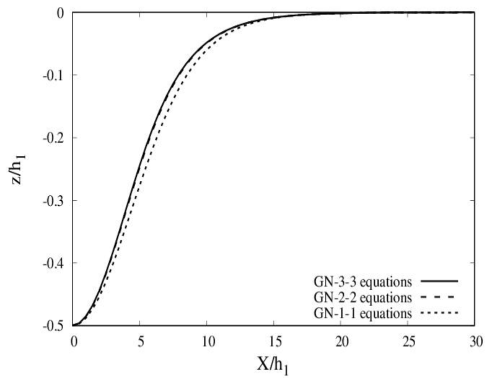
In this section, we will provide some results of the steady solution of internal solitary waves propagating in a linear shear current. The parameters are: g=9.81m/s^2, ρ_1=1000kg/m^3, ρ_2=1001kg/m^3, h_1=1m, h_2=5m. The current strengths $U_1=U_2=U$. Internal solitary-wave amplitude is H/h_1=-0.5. The linear wave speed $c_0=\sqrt{\dfrac{gh_1h_2(\rho_2-\rho_1)}{\rho_1h_2+\rho_2h_1}}\approx 0.0904\text{m/s}$.

4.1 Wave profile

We firstly consider the wave profile when the current strength U=0. Fig. 2(a) shows the wave profiles obtained by the GN-1-1 equations and the strongly nonlinear equations of Choi (2006). We find that the two results show good agreement. Fig. 2(b) shows the wave profiles obtained by different level GN equations. we find that the GN-2-2 results, rather than the GN-1-1 results, are the converged GN results for this case.



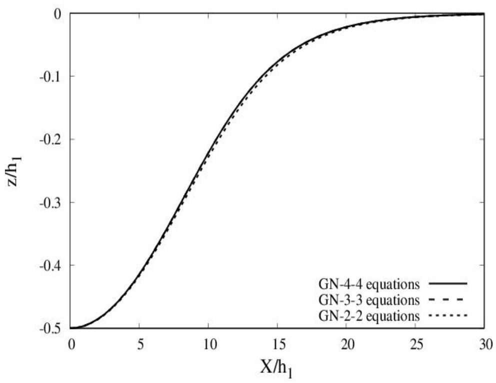
(a) GN-1-1 results



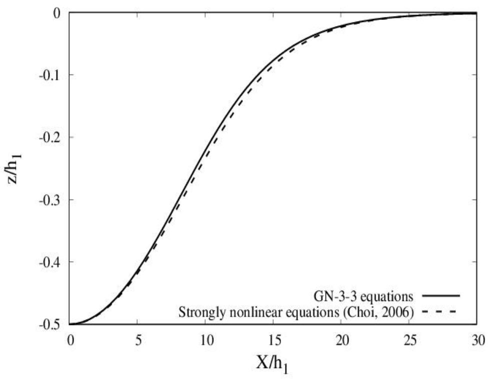
(b) Higher level GN results

Fig. 2 Profiles of the internal solitary wave, current strength *U*=0

Next, we show the wave profiles when the current strength $U/(gh_1)^{1/2}$=0.01 (e.g., U/c_0≈0.3465). Fig. 3(a) shows the different level GN results for the wave profile. After comparison, We find that the GN-3-3 results are the converged GN results. Fig. 3(b) shows the converged GN results and the results of Choi (2006). Some differences are found. The converged GN results maybe more accurate.



(a) GN self-convergence test

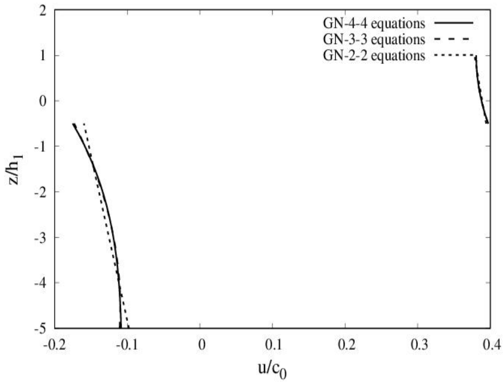


(b) Comparison between the GN results and results of Choi (2006)

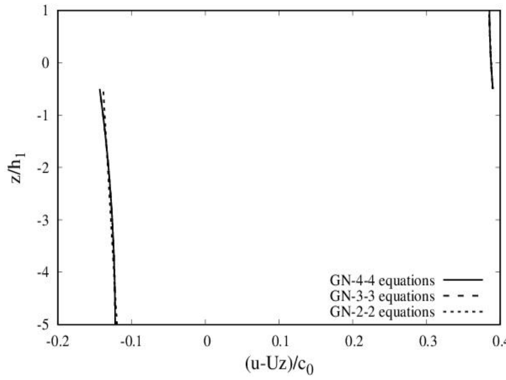
Fig. 3 Profiles of the internal solitary wave, current strength $U/(gh_1)^{1/2}$=0.01

4.2 Velocity field

Here we provide the results of the horizontal velocity along the water column at wave crest for the internal solitary wave with H/h_1=-0.5. Fig. 4(a) shows the results for the irrotational case (U=0). We find that the GN-3-3 equations can be used to obtain the converged results on the velocity distribution. Fig. 4(b) shows the results for the following-current case ($U/(gh_1)^{1/2}$=0.01). We find that the GN-3-3 results are also the converged GN results.



(a) *U*=0



(b) $U/(gh_1)^{1/2}$=0.01

Fig. 4 Horizontal velocity along the water column at wave crest

4.3 Wave speed

We consider the relationship between the wave speed and linear shear current strength. The results are shown in Fig. 5. After comparison, we find that the GN-2-2 results are the converged GN results. Some differences are found between the GN-2-2 results and the results of Choi (2006).

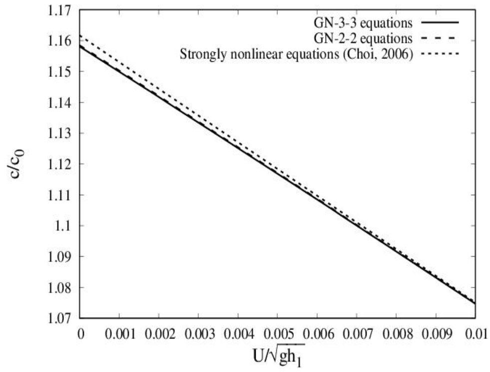


Fig. 5 Relationship between the wave speed and linear shear current strength

More results of cases will be presented at the workshop.

5 CONCLUSIONS

The steady solution of the internal solitary waves in linear shear current is studied. We find that for the irrotational case (U=0), the GN-1-1 results show good agreement with the results obtained by the strongly nonlinear equations of Choi (2006) on wave profile. However, the GN-1-1 results are not the converged GN results. After having the GN self-convergence test, the GN results on the wave profile, velocity field and wave speed are obtained. Meanwhile, for the following-current case ($U/(gh_1)^{1/2}$=0.01), the GN results maybe more accurate due to the fact that we have made the convergence test by using higher level GN equations.

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