The Uniform Motion of an External Load along the Edge of the Ice Cover

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1 INTRODUCTION

The problem about uniform motion of a load has been thoroughly studied for the homogeneous ice sheet that covers the water surface completely. To our knowledge, this problem was first solved by Dotsenko (1976). This solution is also presented in Cherkesov's book (1980). Many studies on this topic were summarized by Squire *et al.* (1996). In the case of inhomogeneous ice cover, there are only some solutions of particular problems. The examples of solutions for a moving load in the case of bounded ice cover can be found in the book by Zhestkaya and Kozin (2003). The basic equations of motion of ice plate and fluid were numerically solved using the finite-element method. The behavior of a very large rectangular elastic plate, which simulates a floating airport during the takeoff and landing of the aircraft, was studied by Kashiwagi (2014). The problem of ice deflection due to a moving load was solved by Brocklehurst (2012) in linear and nonlinear formulations for a semi-infinite ice plate clamped to a vertical wall. Hydroelastic waves caused by a load moving along a frozen channel were studied by Shishmarev *et al.* (2016)

In this paper, we present the solution of a steady three-dimensional problem for flexural-gravity waves generated by a local pressure distribution moving with uniform speed along the rectilinear edge of semi-infinite ice sheet. This load simulates the air-cushion vehicle (ACV). Three configurations are considered: (i) the surface of the fluid is free outside of ice sheet; (ii) two semi-infinite ice sheets (may be of different thickness) divided by a crack with free edges; (iii) the fluid is bounded by a rigid vertical wall and the edge of an ice cover may be both free or clamped. The problem is formulated within linear hydroelastic theory. The fluid is assumed to be inviscid and incompressible and its motion is potential. The ice sheet is treated as an elastic thin plate using the Kirchhoff-Love model. The vertical displacements of ice sheet and free surface are determined at different speeds of ACV, as well as the forces acting on it in horizontal directions.

2 MATHEMATICAL FORMULATION

Let us consider the statement of the problem in the most complicated configuration (ii). The results for the cases (i) and (iii) are briefly presented by Sturova and Tkacheva (2017). Two semi-infinite elastic plates of thicknesses h_1 and h_2 float on water of depth H, and the edges of the plates are free. The plate drafts are ignored. The pressure distribution $P(x, y)$ moves with constant speed U along the rectilinear edge of the right plate. The moving Cartesian coordinate system x, y, z is considered with the x-axis passing through the center of the pressure region perpendicular to the edge of the plate, the y-axis is directed along the crack and the z-axis is directed vertically upwards.

The boundary-value problem for the velocity potential $\varphi(x, y, z, t)$ and deflection of ice sheet $w(x, y, t)$ can be written as

$$\Delta_3\varphi = 0 \quad (-\infty < x, y < \infty,\ -H \le z \le 0),\quad \Delta_3 \equiv \Delta_2 + \partial^2/\partial z^2,\quad \Delta_2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2\ , \tag{1}$$

$$D_n\Delta_2^2 w + \rho h_n U^2 \partial^2 w/\partial y^2 + g\rho_0 w - \rho_0 U \partial\varphi/\partial y = -\mathcal{H}(x)P(x, y) \quad (z = 0),\quad (n = 1, 2), \tag{2}$$

$$\partial\varphi/\partial z = -U\partial w/\partial y \quad (z = 0), \qquad \partial\varphi/\partial z = 0 \quad (z = -H). \tag{3}$$

Here $D_n = Eh_n^3/[12(1-\nu^2)]$; $E,\ \nu,\ \rho$ are Young's modulus, Poisson's ratio and the density of ice sheets, respectively; ρ_0 is the fluid density; g is the acceleration due to gravity; $\mathcal{H}(x)$ is the Heaviside function; $n = 1$ at $x < 0$ and $n = 2$ at $x > 0$. The boundary conditions for plates with free edges have the form

$$\Big(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\Big)w(\pm 0, y) = 0,\quad \frac{\partial}{\partial x}\Big[\frac{\partial^2}{\partial x^2} + (2-\nu)\frac{\partial^2}{\partial y^2}\Big]w(\pm 0, y) = 0. \tag{4}$$

For wave motion the decaying conditions should be satisfied far from the pressure region.

For $h_1 = 0$, we have the configuration (i) in which the fluid is bounded by the free surface at $x < 0$. For configuration (iii), the fluid is restricted at the left by the rigid wall: $\partial\varphi/\partial x = 0$ at $x = 0$. The edge of ice sheet can be free or frozen to the fixed vertical structure, then $w = \partial w/\partial x = 0 \quad (x = 0)$.

We restrict our consideration to the constant pressure distribution in the rectangular planform:

$$P(x, y) = \begin{cases} P_0 = const & (|x - x_0| \le a,\ |y| \le b,\ x_0 > a), \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The forces F_x (side force) and F_y (wave resistance) acting on ACV and its non-dimensional values A_x, A_y are determined by formulae

$$(F_x, F_y) = -P_0 \int_{-b}^{b} \int_{x_0-a}^{x_0+a} \Big(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\Big) dx dy, \quad (A_x, A_y) = -\frac{g\rho_0}{2aP_0^2}(F_x, F_y). \tag{6}$$

3 METHOD OF SOLUTION

We describe briefly the solution of problem (1)∼(4) by the Wiener-Hopf technique. The dimensionless variables and parameters are introduced

$$(x', y', z', a', b', x_0') = \frac{1}{H}(x, y, z, a, b, x_0),\ \beta_n = \frac{D_n}{\rho_0 g H^4},\ F = \frac{U}{\sqrt{gH}},\ \sigma_n = \frac{\rho h_n}{\rho_0 H},\ P_0' = \frac{P_0}{\rho_0 g H}.$$

Below, the primes are omitted. We will seek the velocity potential and the displacement in the form $\varphi = UH\phi(x, y, z), \quad w = HW(x, y)$.

We use the Fourier transform to the variables x and y in the form

$$\Phi^-(\alpha, s, z) = \int_{-\infty}^{\infty} \mathrm{e}^{-isy} dy \int_{-\infty}^{0} \phi(x, y, z)\mathrm{e}^{i\alpha x} dx, \quad \Phi^+(\alpha, s, z) = \int_{-\infty}^{\infty} \mathrm{e}^{-isy} dy \int_{0}^{\infty} \phi(x, y, z)\mathrm{e}^{i\alpha x} dx.$$

From the Laplace equation (1) and no-flux bottom condition (3), we have

$$\Phi(\alpha, s, z) = \Phi^- + \Phi^+ = C(\alpha, s)Z(\alpha, s, z), \quad Z = \cosh[(z + 1)\sqrt{\alpha^2 + s^2}]/\cosh\sqrt{\alpha^2 + s^2}, \tag{7}$$

where $C(\alpha, s)$ is unknown function. We introduce the functions $G_n^{\pm}(\alpha, s)$ in the following manner:

$$G_n^- = \int_{-\infty}^{\infty} \mathrm{e}^{-isy} dy \int_{-\infty}^{0} \Big[\Big(\beta_n\Delta_2^2 + 1 + \sigma_n F^2 \frac{\partial^2}{\partial y^2}\Big)\frac{\partial\phi}{\partial z} + F^2\frac{\partial^2\phi}{\partial y^2}\Big]_{z=0} \mathrm{e}^{i\alpha x} dx,$$

$$G_n^+ = \int_{-\infty}^{\infty} \mathrm{e}^{-isy} dy \int_{0}^{\infty} \Big[\Big(\beta_n\Delta_2^2 + 1 + \sigma_n F^2 \frac{\partial^2}{\partial y^2}\Big)\frac{\partial\phi}{\partial z} + F^2\frac{\partial^2\phi}{\partial y^2}\Big]_{z=0} \mathrm{e}^{i\alpha x} dx.$$

The functions with the superscripts $+/-$ are analytical on α in the upper/lower half-plane, respectively. From boundary conditions (2), we have

$$G_1^-(\alpha, s) \equiv 0, \quad G_2^+(\alpha, s) = isQ(\alpha, s), \quad Q(\alpha, s) = 4P_0\mathrm{e}^{i\alpha x_0}\sin(\alpha a)\sin(sb)/(\alpha s), \tag{8}$$

where $Q(\alpha, s)$ is the Fourier-transform of the function $P(x, y)$ in (5). Using (7), we can write

$$G_1(\alpha, s) = G_1^- + G_1^+ = C(\alpha, s)K_1(\alpha, s), \quad G_2(\alpha, s) = G_2^- + G_2^+ = C(\alpha, s)K_2(\alpha, s), \tag{9}$$

where $K_n(\alpha, s)$ $(n = 1, 2)$ are the dispersion functions for the flexural-gravity waves in a moving coordinate system: $K_n(\alpha, s) = [\beta_n(\alpha^2 + s^2)^2 + 1 - \sigma_n F^2 s^2]\sqrt{\alpha^2 + s^2}\tanh\sqrt{\alpha^2 + s^2} - F^2 s^2$.

It is known that the dispersion relation for the flexural-gravity waves $\mathcal{K}_1(\gamma) \equiv (\beta_1\gamma^4 + 1 - \sigma_1 F^2 s^2)\gamma\tanh\gamma - F^2 s^2 = 0$ has two real roots $\pm\gamma_0$, four complex roots $\pm\gamma_{-1}$, $\pm\gamma_{-2}$, $\gamma_{-2} = -\bar{\gamma}_{-1}$ (the bar denotes complex conjugation), and the countable set of imaginary roots $\pm\gamma_m$, $m = 1, 2, ...$. Similarly, the second relation $\mathcal{K}_2(\mu) \equiv (\beta_2\mu^4 + 1 - \sigma_2 F^2 s^2)\mu\tanh\mu - F^2 s^2 = 0$ has the roots μ_m $(m =$

$-2,-1,0,...)$. Then the roots of the dispersion relations $K_n(\alpha,s)=0$ are $\chi_m=\sqrt{\gamma_m^2-s^2}$ $(n=1)$ and $\alpha_m=\sqrt{\mu_m^2-s^2}$ $(n=2)$. We will take these values in the upper half-plane.

From relations (8) and (9), we obtain $G_2^-(\alpha,s)+isQ(\alpha,s)=G_1^+(\alpha,s)K(\alpha,s)$, $K(\alpha,s)=K_2(\alpha,s)/K_1(\alpha,s)$. In accordance with the Wiener-Hopf technique, we factorize the function $K(\alpha,s)$

$$K(\alpha,s)=K^-(\alpha,s)K^+(\alpha,s),\quad K^\pm(\alpha,s)=\prod_{j=-2}^{\infty}(\alpha\pm\alpha_j)\gamma_j/[\mu_j(\alpha\pm\chi_j)],$$

where $K^\pm$ are analytical in the upper/lower parts of the complex plane α, respectively. After some algebra we obtain the equation

$$G_2^-(\alpha,s)/K^-(\alpha,s)+2P_0\sin(sb)L^-(\alpha,s)=G_1^+(\alpha,s)K^+(\alpha,s)-2P_0\sin(sb)L^+(\alpha,s),\qquad(10)$$

$$L^\pm(\alpha,s)=\pm\frac{1}{2\pi i}\int_{-\infty\mp i\lambda}^{\infty\mp i\lambda}\frac{\psi(\zeta)d\zeta}{K^-(\zeta,s)(\zeta-\alpha)},\quad \psi(\alpha)=\frac{1}{\alpha}[\mathrm{e}^{i\alpha(x_0+a)}-\mathrm{e}^{i\alpha(x_0-a)}].$$

The functions on the left-hand and right-hand sides of Eq. (10) are analytical in the lower and upper parts of the complex plane α, respectively. Then we have analytical function over the entire complex plane α. By Liouville's theorem, this function is a polynomial. The degree of the polynomial is determined by the behavior of this function as $|\alpha|\to\infty$ and is equal to three, *i.e.*

$$G_1^+(\alpha,s)K^+(\alpha,s)-2P_0\sin(sb)L^+(\alpha,s)=2P_0\sin(sb)\sum_{k=0}^{3}a_k(s)\alpha^k,$$

where $a_k(s)$ are unknown functions which are defined from edge conditions (4).

The deflection of the plates is determined by performing inverse Fourier transform:
at $x<0$

$$W(x,y)=-\frac{P_0}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{isy}\sin(sb)}{s}\sum_{j=-2}^{\infty}\frac{\gamma_j\mathrm{e}^{-i\chi_jx}\tanh\gamma_j}{K^+(\chi_j,s)K_1'(\chi_j,s)}\Big[\sum_{k=0}^{3}a_k(s)\chi_j^k+L^+(\chi_j,s)\Big]ds,\qquad(11)$$

at $x>0$

$$W(x,y)=-\frac{P_0}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{isy}\sin(sb)}{s}[S(x,s)+\Lambda(x,s)]ds,\ \Lambda=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi(\alpha)\sqrt{\alpha^2+s^2}\mathrm{e}^{-i\alpha x}\tanh\sqrt{\alpha^2+s^2}}{K_2(\alpha,s)}d\alpha,$$

$$S=\sum_{j=-2}^{\infty}\frac{K^+(\alpha_j,s)}{K_2'(\alpha_j,s)}\mu_j\mathrm{e}^{i\alpha_jx}\tanh\mu_j\Big[\sum_{k=0}^{3}(-1)^ka_k(s)\alpha_j^k-L^-(-\alpha_j,s)\Big],\qquad(12)$$

where the prime denotes the partial derivative of a function with respect to its first variable. The integrand functions in (11), (12) decay exponentially as $|s|\to\infty$.

For each elastic plate there is a minimum phase velocity of the flexural-gravity waves c_n^* $(n=1,2)$ (see, *e.g.* Squire *et al.*, 1996). If $U<c_2^*$, then $\mu_0(s)<|s|$ for any values of the parameter s, and the real roots of dispersion relation $K_2(\alpha,s)=0$ are absent. If $U<c_1^*$, then this is true for another plate, and the real roots of dispersion relation $K_1(\alpha,s)=0$ are absent. At $U<\min\{c_1^*,c_2^*\}$, wave motions in the plates are not excited and only local disturbances are observed in the region of load. If $c_2^*<U<\sqrt{gH}$, then there are two values s_2^- and s_2^+ such that $\mu_0(s)>|s|$ at $s_2^-<|s|<s_2^+$. A similar statement holds for another plate: at $c_1^*<U<\sqrt{gH}$, then there are two values s_1^- and s_1^+ such that $\gamma_0(s)>|s|$ at $s_1^-<|s|<s_1^+$. With increasing speed of the load, the value s_n^- decreases, and the value s_n^+ $(n=1,2)$ increases. At a speed greater than the long-wave limit, *i.e.* $U\geq\sqrt{gH}$, we have $s_1^-=s_2^-=0$.

For the identical plates $(h_1=h_2)$ with a crack, the solution can be obtained in an explicit form. In this case $K_1(\alpha,s)=K_2(\alpha,s)$, $K(\alpha,s)\equiv1$, $K^\pm(\alpha,s)\equiv1$, $L^-(\alpha,s)=0$. The coefficients $a_k(s)$ $(k=0,1,2,3)$ are equal to

$$a_0(s)=\nu s^2a_2(s),\quad a_1(s)=(2-\nu)s^2a_3(s),$$

$$a_2(s)\Omega(s) = -2P_0 \sin(sb) \sum_{j=-2}^{\infty} \frac{(\alpha_j^2 + \nu s^2)\psi(\alpha_j)\mu_j \tanh \mu_j}{K_2'(\alpha_j, s)}, \ \Omega(s) = \sum_{j=-2}^{\infty} \frac{(\alpha_j^2 + \nu s^2)^2 \mu_j \tanh \mu_j}{K_2'(\alpha_j, s)},$$

$$a_3(s) \sum_{j=-2}^{\infty} \frac{\alpha_j^2[\alpha_j^2 + (2-\nu)s^2]^2 \mu_j \tanh \mu_j}{K_2'(\alpha_j, s)} = -2P_0 \sin(sb) \sum_{j=-2}^{\infty} \frac{\alpha_j[\alpha_j^2 + (2-\nu)s^2]\psi(\alpha_j)\mu_j \tanh \mu_j}{K_2'(\alpha_j, s)}.$$

At $U > c^*$, there are two values s_*^- and s_*^+ for which $\Omega(s) = 0$. The values s_*^- and s_*^+ are extremely close to the values s^- and s^+ ($s_*^- < s^-$, $s_*^+ > s^+$). The residues of the integrand at these points determine the edge waveguide modes with wave numbers s_*^- and s_*^+. The mode with s_*^+ propagates in front of the load, but the mode with s_*^- propagates behind the load. For configuration (iii), the solution is also constructed in explicit form using the reflection method. The most large amplitude of the edge mode is for configuration (iii) with free edge.

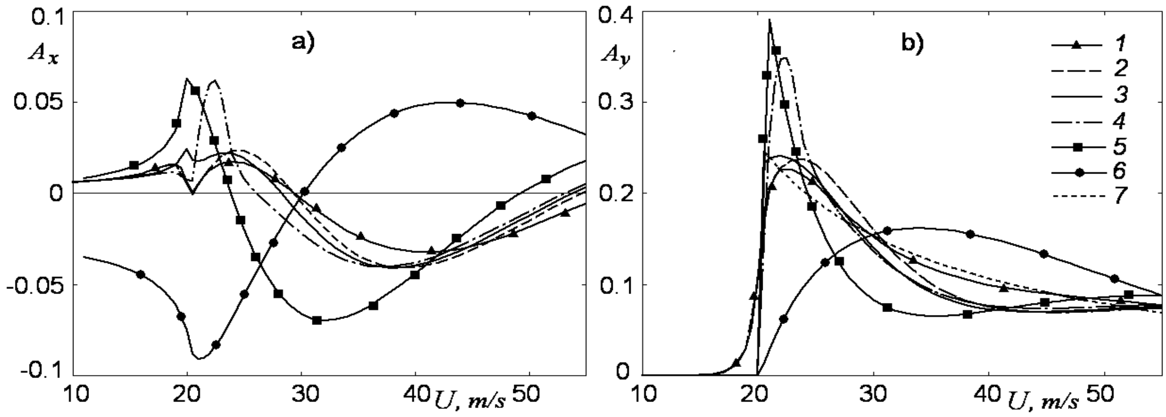

Figure. 1.

Non-dimensional values of wave forces acting on moving vehicle are presented in Fig. 1 as functions of the load speed. The following input data are used: $E = 5\ GPa$, $\rho = 900\ kg/m^3$, $\nu = 1/3$, $h_2 = 2\ m$, $\rho_0 = 10^3\ kg/m^3$, $a = 10\ m$, $b = 20\ m$, $x_0 = 50\ m$, $H = 350\ m$, $P_0 = 10^3\ Pa$. The minimum phase velocity of the flexural-gravity waves for these parameters is equal to 20.14 m/s. Fig. 1(a,b) shows the values A_x and A_y in (6), respectively. Curves *1* correspond to configuration (i). Curves *2-4* show the results for configuration (ii) at $h_1 = 1,\ 2,\ 3\ m$, respectively. Curves *5,6* correspond to configuration (iii) with free and clamped edge, respectively. The wave resistance for infinite ice sheet is given by curve *7*, the side force in this case is identically equal to zero. More detailed numerical results will be presented at the Workshop.

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