Plate arrays as a water wave metamaterial

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1 Introduction

The term metamaterial is used to describe a medium which exhibits properties not normally associated with naturally-occurring materials. Typically, metamaterials achieve this by possessing a microstructure whose lengthscale is significantly smaller than the characteristic lengthscale over which the underlying field varies. The effect of the microscale on the macroscale gives rise to an effective medium whose properties are normally established by homogenisation methods or by direct numerical simulation. Metamaterial science emerged in tandem with discoveries made primarily in the physical disciplines of optics and electromagnetics. Most notably these include negative refraction, perfect lensing and invisibility cloaking. In each case the properties required of the metamaterial are prescribed by a specific design outcome as exemplified by so-called transformation media methods used for designing cloaks. The microstructure used in most metamaterials is built upon two- and three-dimensional periodic lattices of scatterers or resonators (e.g. Smith *et al* (2012)).

In water waves, a metamaterial can be formed by any medium with structural elements smaller than the anticipated range of wavelengths. In this paper we consider a particular embodiment of a water wave metamaterial in which the medium is formed by closely-spaced arrays of thin vertical plates extending through the depth.

We demonstrate the unusual properties of this simple but novel metamaterial using three examples each involving the scattering of plane waves on a fluid of constant depth. The approach taken to the first problem is similar to work by Kaji & Okazaki (1970) who considered sound propagation in the presence of mean flow through a linear array of turbine blades.

2 Scattering by a plate-array metamaterial of finite width

In this problem the metamaterial medium occupies $|y| < b$, $-\infty < x < \infty$ and is comprised of a periodic array of thin plates rotated through a clockwise angle δ and separated by a perpendicular distance d from neighbouring plates in the array (plan view in Fig. 1(a)). The plates are of length $2L = 2b\cos\delta$. They extend uniformly through the fluid depth, h, allowing us to remove the depth dependence, z along with a harmonic time dependence of angular frequency ω from the velocity potential $\Re\{\phi(x,y)\cosh k(z+h)\mathrm{e}^{-\mathrm{i}\omega t}\}$ where $k\tanh kh = \omega^2/g$. The wavefunction $\phi(x,y)$ satisfies

$$(\nabla^2 + k^2)\phi = 0, \qquad \text{in the fluid.} \tag{1}$$

A plane wave is incident from $y < -b$ and propagates in the direction θ_0 measured clockwise from the positive y-axis and so

$$\phi(x,y) \sim \begin{cases} \mathrm{e}^{\mathrm{i}\alpha_0 x}\mathrm{e}^{\mathrm{i}\beta_0 y} + R\mathrm{e}^{\mathrm{i}\alpha_0 x}\mathrm{e}^{-\mathrm{i}\beta_0 y}, & \text{as } y \to -\infty \\ T\mathrm{e}^{\mathrm{i}\alpha_0 x}\mathrm{e}^{\mathrm{i}\beta_0 y}, & \text{as } y \to \infty \end{cases} \tag{2}$$

where $\alpha_0 = k\sin\theta_0$, $\beta_0 = k\cos\theta_0$ and R and T are reflection and transmission coefficients (assuming kd is small enough that no further diffraction modes exist). Neumann conditions apply on each side of the plates and ϕ is inverse square-root singular at the ends of the plates.

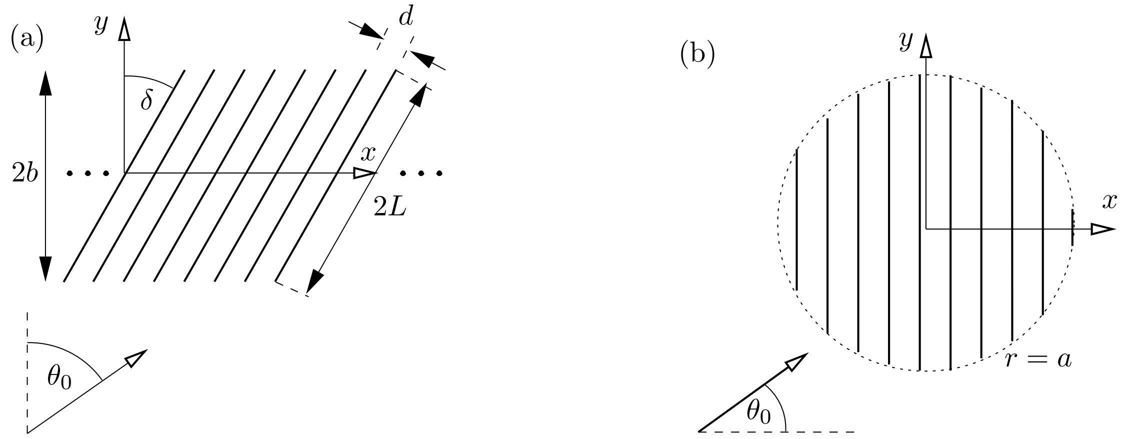

Figure 1: (a) A parallel-plate array metamaterial of finite width and (b) a circular parallel-plate array metamaterial cylinder.

Solutions to the full microstructured problem described above can be sought using Bloch-Floquet theory to relate $\phi(x, y)$ everywhere to the value it takes in a fundamental cell of the array of width d. One can then derive an integral equation over the plate within that cell whose solution can be numerically calculated. Evidently this is a complicated approach.

Alternatively we can take advantage of the underlying assumption that the separation between plates is small both with respect to the wavelength and the plate length ($kd \ll 1$, $d/L \ll 1$) and replace the plate/fluid microstructure in $|y| < b$ with an effective medium in which the potential $\phi(x, y) \approx \Phi(X, Y)$ with $y = Y \cos\delta$, $x = Y \sin\delta + X$ satisfies

$$\left(\frac{\partial^2}{\partial Y^2} + k^2\right)\Phi = 0, \qquad |Y| < L, \ -\infty < X < \infty. \tag{3}$$

and whose general solution is

$$\Phi(X, Y) = c(X)\mathrm{e}^{\mathrm{i}kY} + d(X)\mathrm{e}^{-\mathrm{i}kY}. \tag{4}$$

We also note that the representation (2) – previously asymptotic – is now exact for $y < -b$ and $y > b$. Continuity of the field across the upper and lower boundaries requires

$$\phi(x, -b^-) = \Phi(x - L\sin\delta, -L^+), \qquad \text{and} \qquad \phi(x, b^+) = \Phi(x + L\sin\delta, L^-) \tag{5}$$

and it follows that $c(X) = C\mathrm{e}^{\mathrm{i}\alpha_0 X}$ and $d(X) = D\mathrm{e}^{\mathrm{i}\alpha_0 X}$, for $C, D \in \mathbb{C}$. A second matching condition comes from balancing fluxes across small triangles the edge of the microstructure which leads to

$$\phi_y(x, -b^-) = \cos\delta\,\Phi_Y(x - L\sin\delta, -L^+), \qquad \text{and} \qquad \phi_y(x, b^+) = \cos\delta\,\Phi_Y(x + L\sin\delta, L^-). \tag{6}$$

The four matching conditions (5), (6) allow us to find C, D and

$$R = \frac{(\cos^2\theta_0 - \cos^2\delta)\sin(2kL)\mathrm{e}^{-2\mathrm{i}kL\cos\theta_0\cos\delta}}{(\cos^2\theta_0 + \cos^2\delta)\sin(2kL) + 2\mathrm{i}\cos(2kL)\cos\theta_0\cos\delta} \tag{7}$$

and

$$T = \frac{2\mathrm{i}\cos\theta_0\cos\delta\mathrm{e}^{-2\mathrm{i}kL\cos(\theta_0-\delta)}}{(\cos^2\theta_0 + \cos^2\delta)\sin(2kL) + 2\mathrm{i}\cos(2kL)\cos\theta_0\cos\delta}. \tag{8}$$

Several things are worthy of note. First, the conservation of energy relation, $|R|^2 + |T|^2 = 1$, is easily verified. Writing $R = R(\theta_0, \delta; kL)$, $T = T(\theta_0, \delta; kL)$ helps list further properties as:

$$\text{(i) (symmetry)} \quad R(\theta_0, \delta; kL) = R(-\theta_0, \delta; kL), \quad \text{and} \quad R(\theta_0, \delta; kL) = R(\theta_0, -\delta; kL) \tag{9}$$

with the same relations applying to $|T|$;

$$\text{(ii) (angular transparency)} \quad R(\theta_0, \pm\theta_0, kL) = 0, \qquad \begin{cases} T(\theta_0, \theta_0, kL) = 1, \\ |T(\theta_0, -\theta_0, kL)| = 1 \end{cases} \tag{10}$$

and (wavenumber transparency) $R(\theta_0, \delta, n\pi/2) = 0$ with $|T(\theta_0, \delta, n\pi/2| = 1$;

$$\text{(iii) (reciprocity)} \quad R(\theta_0, \delta, kL) = -R(\delta, \theta_0, kL), \quad \text{and} \quad T(\theta_0, \delta, kL) = T(\delta, \theta_0, kL); \tag{11}$$

$$\text{(iv) (periodicity)} \quad |R(\theta_0, \theta_0, kL + n\pi/2)| = |R(\theta_0, \theta_0, kL)| \quad \text{and} \quad |R(\theta_0, \theta_0, \pi/2 - kL)| = |R(\theta_0, \theta_0, kL)| \tag{12}$$

with the same relations applying to $|T|$.

Amongst the results above, most remarkable is the first result in (ii), that of total transmission for $\theta_0 = -\delta$ when the array is angled backwards against the incident wave direction a property which is wavenumber independent. This property is shared by the solution to the *exact* microstructured problem. Comparisons have been made between the approximate solution presented here based on a homogenised governing equation for the metamaterial plate array and the exact treatment of the microstructured problem and show excellent agreement provided d/L is small enough (roughly less than 0.1).

An illustration of the transparency property is presented in Fig. 2(a,b) which shows the instantaneous wave field for a wave with $k = 1, 2$ propagating at $\theta_0 = 45°$ across an array of width $b = 2$ tilted at $\delta = -45°$. An illustration of wavenumber transparency is shown in Fig. 2(c) where a Gaussian beam (a weighted integral over a range of values of θ_0) is incident on the same array for $kL = 18\pi$. Thus the array acts as a negative refraction medium and a 'metamaterial waveshifter' – e.g. Smith *et al.* (2012).

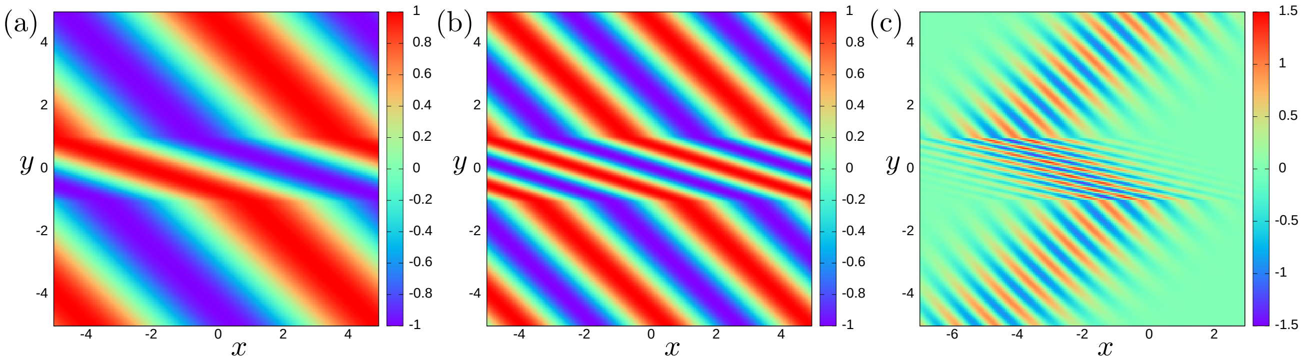

Figure 2: The instantaneous wave field for incident waves propagating at $\theta_0 = 45°$ into an array of plates tilted at $\delta = -45°$ for $k = 1$ (a) and $k = 2$ (b). In (c) a Gaussian beam is incident at $kL/2\pi = 9$.

3 Scattering by a circular metamaterial cylinder

A parallel array of closely-spaced vertical plates extending uniformly throughout the depth are aligned with the y-axis and fill a circular cylinder of radius a centred at the origin. Plane waves propagating at an angle θ_0 w.r.t. to the x-axis are incident on the cylindrical structure (see Fig. 1(b).)

In $r > a$ the general solution satisfying the wave equation (1) is

$$\phi(r, \theta) = \sum_{n=-\infty}^{\infty} \mathrm{i}^n (J_n(kr)\mathrm{e}^{-\mathrm{i}n\theta_0} + a_n H_n(kr))\mathrm{e}^{\mathrm{i}n\theta} \tag{13}$$

where a_n are scattering coefficients, to be determined.

Inside the cylinder the array of plates act as a metamaterial meaning the field satisfies (see (3))

$$\Phi_{yy} + k^2\Phi = 0, \qquad r < a \tag{14}$$

and solutions are

$$\Phi(x, y) = c(x)\mathrm{e}^{\mathrm{i}ky} + d(x)\mathrm{e}^{-\mathrm{i}ky}, \qquad r < a. \tag{15}$$

We expand the two unknown functions of x in terms of Chebychev polynomials and unknown expansion coefficients c_n, d_n

$$c(x) = \sum_{n=0}^{\infty} c_n T_n(x/a), \qquad d(x) = \sum_{n=0}^{\infty} d_n T_n(x/a) \tag{16}$$

a choice which allows simplification of subsequent algebra. For example, we find

$$\Phi(a\cos\theta, a\sin\theta) = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{p=-\infty}^{\infty} (c_n + (-1)^{p+n} d_n)(J_{p+n}(ka) + J_{p-n}(ka))\mathrm{e}^{\mathrm{i}p\theta} \tag{17}$$

which is matched to $\phi(a,\theta)$ from (13). The matching of fluxes across the boundary of the cylinder to the region between the plates gives the condition $\phi_r(a,\theta) = \sin\theta\ \Phi_y(a\cos\theta, a\sin\theta)$. Together the two matching conditions give rise to a pair of infinite systems of equations for for the coefficients a_n $(-\infty < n < \infty)$ and c_n, d_n $(0 \leq n < \infty)$, solved numerically by truncation to finite systems of equations.

When $\theta_0 = \pm\pi/2$ the solution reduces to $a_n = 0$ and the waves pass through the cylinder with no scattering. For other values of θ_0 the metamaterial cylinder interacts with incident waves in a non-trivial way. Fig. 3 illustrates typical results and shows the instantaneous free surface for a cylinder with $a = 1$ and $\theta_0 = 45°$ and $k = 1$, $k = 1.3$, $k = 1.6$. The color scales differ in each plot and the maximum height of the wave field varies from 2.5 to 25 across the three plots. The narrow open-ended channels between the plates act as one-dimensional resonators and their effect on the wave field increases until $ka = \pi/2$ when the diameter of the cylinder matches a half-wavelength of the incident wave and the central channel of the metamaterial cylinder is resonant. Once ka exceeds $\pi/2$ two shorter chords of cylinder are excited to resonance by incident waves as indicated in Fig. 3(c) by the two thin high-amplitude lines across the cylinder. Further results will be presented and discussed at the Workshop.

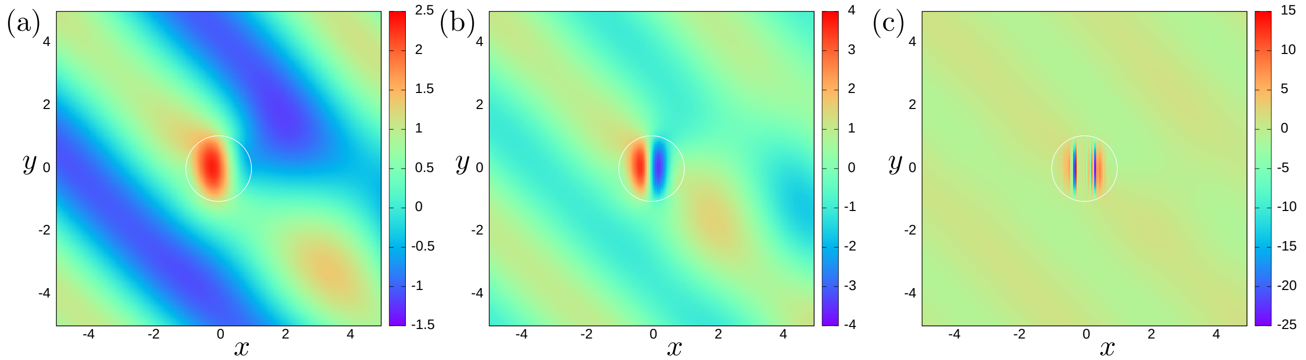

Figure 3: Instantaneous wave field due to incident wave propagating with $\theta_0 = 45°$ on a cylinder with $a = 1$ with: (a) $k = 1$; (b) $k = 1.3$; (c) $k = 1.6$.

4 Reflection of water waves by a metamaterial wall

In a third example, results will also be shown for waves interacting with a wall comprised of a staggered plate array. Here, it will be shown that plane waves of incident angle θ_0 can be totally reflected in the same direction as the incoming wave.

References

1. S. Kaji & K. Okazaki (1970) Generation of Sound by Rotor-Stator Interaction, *Journal of Sound and Vibration* **13**(3), 281–307.
2. M.J.A. Smith, R.C. McPhedran, C.G. Poulton & M.H. Meylan (2012) Negative refraction and dispersion phenomena in platonic clusters. *Waves in Random and Complex Media*, **22** 1–24.