Rainbow trapping of water waves

M. A. Peter[1], L. G. Bennetts[2] and R. V. Craster[3]

[1]Institute of Mathematics, University of Augsburg, 86135 Augsburg, Germany
[2]School of Mathematical Sciences, University of Adelaide, Adelaide, SA 5005, Australia.
[3]Department of Mathematics, Imperial College London, South Kensington Campus, UK
e-mail address: malte.peter@math.uni-augsburg.de

1 Introduction

At the workshop in Athens in 2011, Peter & Meylan (2011) described how band-structure computations for periodic lattices (crystals) of arbitrary bodies can be performed using interaction theories and how this observation can be used to compute band structures for periodic lattices of arbitrary scatterers of water waves. Moreover, they discussed how truncated (i.e. finite number) periodic arrangements of bottom-mounted cylinders can be used to focus water waves, where the arrangement of cylinders acts as a lens of effective negative refractive index.

On the other hand, in acoustics, Cebrecos *et al.* (2014) showed that a so-called chirped crystal of cylinders, with lattice spacing decreasing in the direction of the incident wave, can be used to amplify (acoustic) pressure in certain regions. They showed this experimentally but also argued that the effect is due to the incident wave reaching a region inside the grating where its effective group velocity vanishes so that the incident energy accumulates. Band-structure calculations were used to support this argument, and, clearly, only mild lattice spacing variations are allowed in order for each cylinder to 'feel' as if being one member of a regular crystal. Different frequencies are amplified in different regions, leading to the phenomenon being referred to as rainbow trapping.

Here, we provide, to our knowledge, the first demonstration of rainbow trapping of water waves. Specifically, we show how the energy of a normal incident wave can be amplified in truncated chirped crystals of rigid bottom-mounted cylinders, with N columns and M rows. Figure 1 illustrates the form of crystals considered, which are analogous to those of Cebrecos *et al.* (2014) but with increasing spacing in order to trap relevant frequencies. This idea could be used to increase the efficiency of power take-off devices in the ocean.

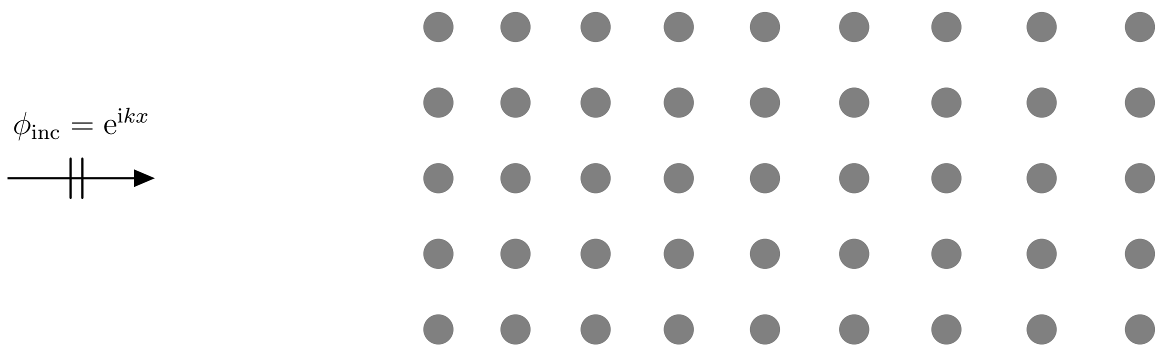

Figure 1: Schematic of a chirped crystal with $N = 8$ columns and $M = 5$ rows (top view).

2 Statement of the problem and solution method

We consider water-wave scattering by an arrangement of (non-overlapping) surface-piercing rigid bottom-mounted circular cylinders. The equations of motion for the water are derived from the linearised inviscid theory assuming irrotational motion. Restricting to time-harmonic motion with radian frequency ω (which is the spectral parameter), the velocity potential Φ can be expressed as the real part of a complex quantity, $\Phi(\mathbf{y},t) = \mathbb{R}\mathrm{e}\,\{\phi(\mathbf{y})\mathrm{e}^{-\mathrm{i}\omega t}\}$, where $\mathbf{y} = (x, y, z)$ denotes a point in the water. The notation $\mathbf{x} = (x, y)$ denotes a point on the undisturbed water surface, assumed at $z = 0$, i.e. $\mathbf{x} = (x, y, 0)$.

Writing $\alpha = \omega^2/g$, where $g \approx 9.81\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity, the potential ϕ has to satisfy the standard boundary-value problem

$$-\Delta\phi = 0, \qquad \mathbf{y} \in D, \tag{1}$$

$$\frac{\partial\phi}{\partial z} = \alpha\phi, \qquad \mathbf{x} \in \Gamma^{\mathrm{f}}, \tag{2}$$

$$\frac{\partial\phi}{\partial z} = 0, \qquad \mathbf{y} \in D,\ z = -d, \tag{3}$$

where $D = (\mathbb{R}^2 \times (-d, 0)) \setminus \bigcup_j \overline{\Delta}_j$ is the domain occupied by the water (Δ_j is the jth cylinder) and Γ^{f} is the free water surface. The normal derivative of the potential on the cylinder surfaces is required to vanish. For future reference, we note that the positive wavenumber k is related to α by the dispersion relation $\alpha = k \tanh kd$. Owing to the constant cross-section of the cylinders, the depth dependence can be factored out of the problem, and restricting to incident waves of propagating nature, outside of the escribed cylinder of a scatterer any solution can be written as

$$\phi(x, y, z) = f_0(z)\phi_0(x, y), \quad \text{where} \quad f_0(z) = \cosh k(z + d)\,\mathrm{sech}(kd) \tag{4}$$

is the vertical eigenfunction and ϕ_0 has to satisfy the corresponding Helmholtz equation

$$-\Delta\phi_0(x, y) = k_0^2\phi_0(x, y), \quad \text{where} \quad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2. \tag{5}$$

The potential can be solved for efficiently using a modified version of the recursive algorithm developed by Montiel *et al.* (2016) and Bennetts *et al.* (2017).

2.1 The periodic problem

Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be two (two-dimensional) vectors that span the lattice: that is every translation between the mean-centre position of bodies in the horizontal plane has the form of a *lattice vector* $\mathbf{R} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2$, where $m_1, m_2 \in \mathbb{Z}$. The corresponding *reciprocal lattice vectors* $\mathbf{K}$ satisfy $\mathbf{K} \cdot \mathbf{R} = 2\pi p$, where $p \in \mathbb{Z}$. If the reciprocal lattice vectors are written as $\mathbf{K} = n_1\mathbf{b}_1 + n_2\mathbf{b}_2$ for $n_1, n_2 \in \mathbb{Z}$, the relation of the reciprocal lattice vectors is satisfied provided that $\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi\delta_{ij}$, where δ_{ij} is the Kronecker delta. Bloch's theorem justifies looking for solutions of the form

$$\phi(\mathbf{y} + (\mathbf{R}, 0)) = \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{R}}\phi(\mathbf{y}), \tag{6}$$

for all lattice vectors $\mathbf{R}$. The real part of $\mathbf{q}$ measures the change in the phase, while the imaginary part encodes the change in amplitude.

If $\phi(\mathbf{y}; \mathbf{q})$ is a solution then so is $\phi(\mathbf{y}; \mathbf{q} + \mathbf{K})$, meaning we can restrict attention to the *first Brillouin zone* $\{\mathbf{q}\,|\,\mathbb{R}\mathrm{e}\,q_1 \in (-\pi/L, \pi/L],\ \mathbb{R}\mathrm{e}\,q_2 \in (-\pi/W, \pi/W)\}$. For a specified frequency ω, we solve for possible $\mathbf{q}$ values using the method of McIver (2000) as described in Peter & Meylan (2011).

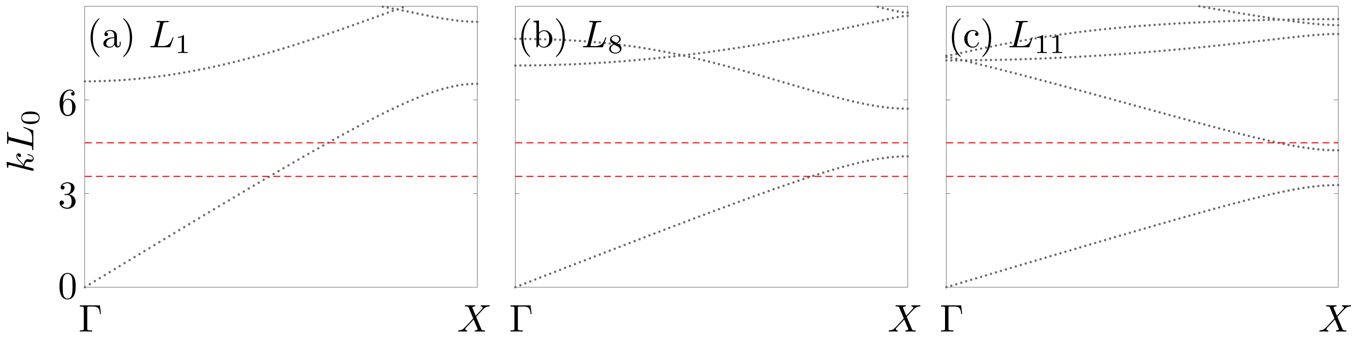

Figure 2: Band diagrams (•) for cell width $W = 0.8L_0$, and cell lengths set as the spacing between the (a) first and second column, (b) eighth and ninth columns, and (c) eleventh and twelfth columns. Nondimensional incident wavenumbers $kL_0 = 3.550$ and 4.625 are superimposed (--).

3 Results

We present results for a plane incident wave, $\phi_{\text{inc}} = \mathrm{e}^{\mathrm{i}kx}$, on a chirped crystal with $M = 6$ rows and $N = 14$ columns, as in Cebrecos *et al.* (2014). The spacing between the rows is constant W, and the spacing between columns n and $n + 1$ is L_n, where

$$L_n = L_0 \mathrm{e}^{-\alpha x_{N-n}} \quad \text{for} \quad n = 1, \ldots, N-1, \tag{7}$$

and x_n is the x-location of column n, with $x_1 = 0$.

Figure 2 shows band diagrams for cell widths W and lengths $L = L_n$, at different positions n along the chirped crystal. As can be seen, a gap opens up as the position moves into the crystal. In particular, non-dimensional wavenumbers $kL_0 = 3.550$ and 4.625 are in the band gap between the eighth and ninth and the eleventh and twelfth columns, respectively. Figure 3 shows corresponding plots of the normalised energy fields, in which the amplification of the incident energy by factors ~ 30 and ~ 20, respectively, can be observed inside the crystal.

4 Conclusions

We showed how an arrangement of surface-piercing bottom-mounted rigid circular cylinders resembling a truncated chirped crystal can be used to amplify incident water-wave energy at a position inside the crystal depending on the frequency of the incident wave (rainbow trapping). The amplification factors were of the order of 30 and 20 for the two examples presented. This idea has the potential of being used to increase the efficiency of power take-off devices in the ocean.

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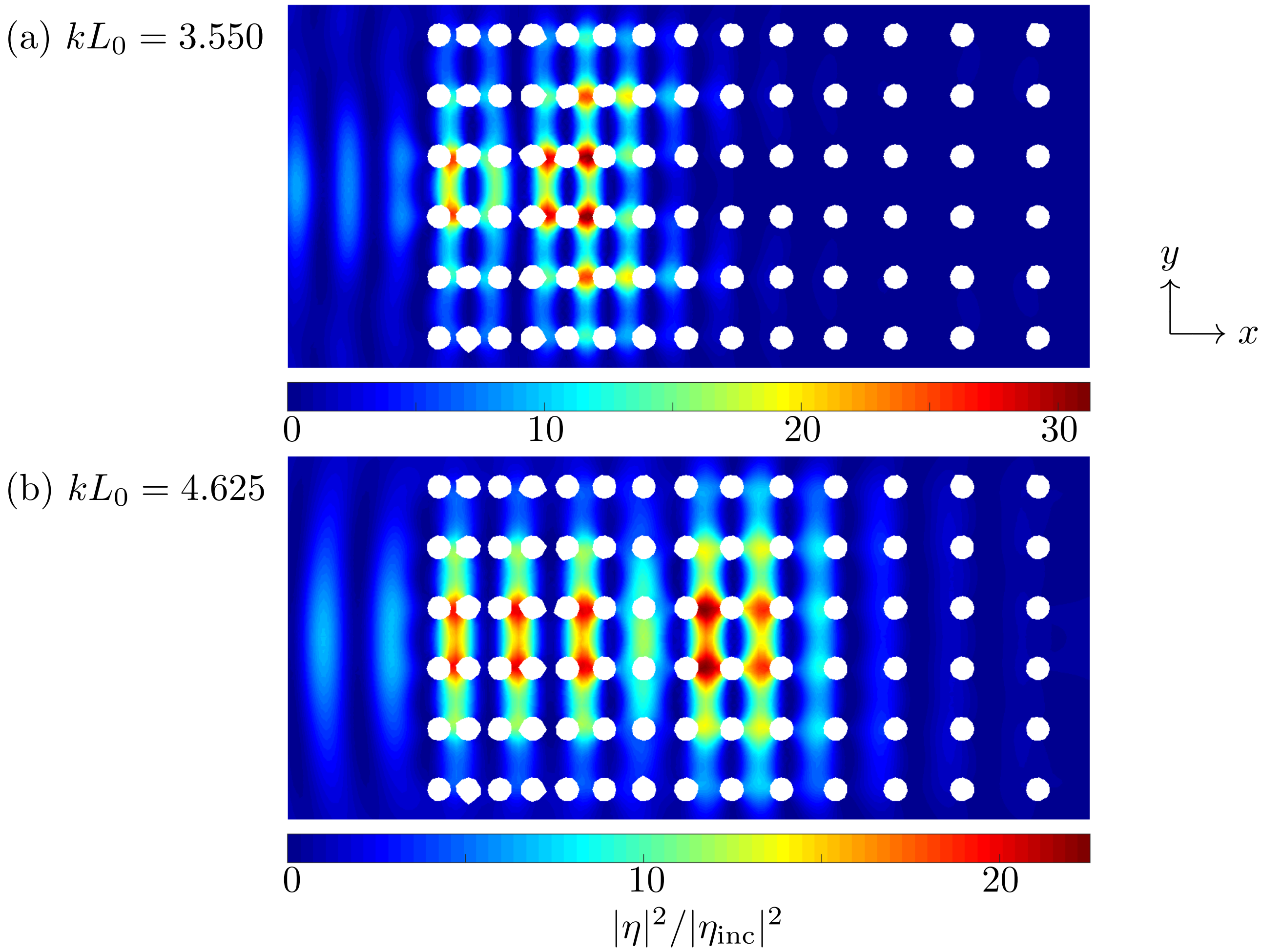

Figure 3: Normalised energy fields for (a) $kL_0 = 3.550$ and (b) $kL_0 = 4.625$.

References

Bennetts, L. G., Peter, M. A. & Montiel, F. 2017 Localisation of Rayleigh–Bloch waves and damping of resonant loads on arrays of vertical cylinders. *J. Fluid Mech.* **813**, 508–527.

Cebrecos, A., Picó, R., Sánchez-Morcillo, V. J., Staliunas, K., Romero-Garcia, V. & Garcia-Raffi, L. M. 2014 Enhancement of sound by soft reflections in exponentially chirped crystals. *AIP Advances* **4**, 124402.

McIver, P. 2000 Water-wave propagation through an infinite array of cylindrical structures. *J. Fluid Mech.* **424**, 101–125.

Montiel, F., Squire, V. A. & Bennetts, L. G. 2016 Attenuation and directional spreading of ocean wave spectra in the marginal ice zone. *J. Fluid Mech.* **790**, 492–522.

Peter, M. A. & Meylan, M. H. 2011 Band structures and band gaps in water-wave scattering by periodic lattices of arbitrary bodies. In *Proc. 26th Int. Workshop on Water Waves and Floating Bodies, Athens, Greece* (ed. S. A. Mavrakos & I. K. Chatjigeorgiou), pp. 141–144.