Boundary-integral relations in the theory of ship motions in regular waves

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Highlight

The classical boundary-integral formulation of potential flow around a ship that travels at a constant speed in regular waves is reconsidered, and a modified formulation that is significantly better suited for accurate numerical evaluation than the classical formulation is given. In the special case of a ship that travels in calm water, the modified boundary-integral formulation obtained here provides an interesting alternative to the formulation of the Neumann-Michell theory given previously.

1. Introduction and basic notation

The flow around a ship hull, of length L, that travels along a straight path, at a constant speed V, through time-harmonic (regular) ambient waves in water of large depth and lateral extent is considered within the usual framework of linear potential flow theory. The flow is observed from a Galilean system of coordinates that advances along the path of the ship at the ship speed V. The encounter frequency of the ambient waves is denoted as ω. The X axis is chosen as the path of the ship and points toward the ship bow. The Z axis is vertical and points upward, and the undisturbed free surface is taken as the plane $Z=0$. The mean wetted hull surface of the ship and its intersection with the undisturbed free-surface plane $Z=0$ are denoted as Σ_H and Γ, which is oriented clockwise when viewed from above the free-surface plane $Z=0$, and Σ_F denotes the undisturbed free surface outside the mean ship waterline Γ. The nondimensional wave frequency f, the Froude number F and the related parameter τ are defined as $f\equiv\omega\sqrt{L/g}$, $F\equiv V/\sqrt{gL}$ and $\tau\equiv fF\equiv V\omega/g$, where g denotes the acceleration of gravity.

The coordinates $\mathbf{x}\equiv(x,y,z\le 0)$ and $\widetilde{\mathbf{x}}\equiv(\widetilde{x},\widetilde{y},\widetilde{z}\le 0)$, used further on, the time t, the flow potential ϕ, velocity $\nabla_{\mathbf{x}}\phi$, pressure p and surface flux q are nondimensional in terms of the length L and the speed V of the ship, the gravitational acceleration g and the water density ρ, as follows:

$$\mathbf{x}\equiv\mathbf{X}/L\ ,\ t\equiv T\sqrt{g/L}\ ,\ \phi\equiv\Phi/(VL)\ ,\ \nabla_{\mathbf{x}}\phi\equiv\nabla_{\mathbf{X}}\Phi/V\ ,\ p\equiv P/(\rho V^2)\ ,\ q\equiv Q/V$$

The unit vector $\mathbf{n}\equiv\mathbf{n}(\mathbf{x})\equiv(n^x,n^y,n^z)$ normal to Σ_H at a point $\mathbf{x}$ of Σ_H points outside the ship (into the water). The unit vector $\mathbf{t}\equiv(t^x,t^y,0)=(n^y,-n^x,0)/\sqrt{1-(n^z)^2}$ tangent to the mean waterline Γ at a point $\mathbf{x}=(x,y,0)$ of Γ points toward the bow or the stern of the ship on the positive half $0\le y$ or the negative half $y\le 0$ of Γ.

2. Generic boundary-value problem

The flow potential $\widehat{\phi}(\mathbf{x},t)$ is expressed as $\widehat{\phi}(\mathbf{x},t)=\mathrm{Re}\,\phi(\mathbf{x})\,e^{-\mathrm{i}f_\epsilon t}$ where $f_\epsilon\equiv f+\mathrm{i}\epsilon$ and $0<\epsilon\ll 1$. This flow potential satisfies the initial conditions $\widehat{\phi}=0$ and $\partial\widehat{\phi}/\partial t=0$ for $t=-\infty$. The spatial component $\phi(\mathbf{x})$ vanishes as $|\mathbf{x}|\to\infty$, and satisfies the Laplace equation

$$\nabla^2\phi\equiv(\partial_x^2+\partial_y^2+\partial_z^2)\,\phi=0 \tag{1a}$$

in the undisturbed flow region D, the linearized boundary condition

$$[\partial_z+(\mathrm{i}f_\epsilon+F\partial_x)^2]\,\phi=(\mathrm{i}\tau+F^2\partial_x)\,p^F-q^F\equiv\pi^F \tag{1b}$$

at the undisturbed free surface Σ_F and the Neumann boundary condition

$$\mathbf{n}\cdot\nabla\phi\equiv\partial\phi/\partial n=q_{_H}\ \text{where}\ q_{_H}\equiv\mathbf{n}\cdot\mathbf{v}_H \tag{1c}$$

at the mean wetted hull surface Σ_H of the ship.

In the generic boundary-value problem (1) considered here, the flux $q_{_H}$ related to the normal component of the velocity $\mathbf{v}_H\equiv\mathbf{V}_H/V$ of the ship is presumed to be given at every point $\mathbf{x}$ of Σ_H in the boundary condition (1c). In the boundary condition (1b), $p^F(x,y)$ and $q^F(x,y)$ represent eventual pressure and flux distributions at the undisturbed free surface Σ_F. The free-surface pressure p^F and flux q^F are also presumed to be specified at every point $(x,y,0)$ of Σ_F. One has $p^F=0$ for most ships, but $p^F\neq 0$ for some types of vessels like hovercrafts and Surface-Effect-Ships. The free-surface flux q^F is considered because it is useful for the Green function associated with the boundary condition (1b).

The special case $F=0$ of the boundary-value problem (1) is a trivial case for which practical solutions exist. The 'forward-speed case' $F\neq 0$ is hugely more complicated. In the special case $f=0$, a practical solution to the Neumann-Kelvin theory proposed by Brard in 1972 and Guevel in 1974 exists [1]. The boundary-value problem (1) is far more complicated in the general case $\tau\neq 0$ than in the special case

$f = 0$. Although useful solution procedures have been reported in the literature, not reviewed here, a fully satisfactory method for solving this difficult important problem remains a difficult goal.

3. Green function

A Green function $G(\mathbf{x}, \widetilde{\mathbf{x}})$ that is associated with the Laplace equation (1a) and the free-surface boundary condition (1b) is now introduced. Specifically, this Green function satisfies the equations

$$\left\{\begin{array}{l}(\partial_x^2+\partial_y^2+\partial_z^2)\,G = \\ \quad \delta(x-\widetilde{x})\,\delta(y-\widetilde{y})\,\delta(z-\widetilde{z}) \text{ in } z<0 \\ \{\partial_z+(\mathrm{i}f_\epsilon-F\partial_x)^2\}\,G=0 \text{ at } z=0\end{array}\right\} \text{ if } \widetilde{z}<0 \qquad \left\{\begin{array}{l}(\partial_x^2+\partial_y^2+\partial_z^2)\,G=0 \text{ in } z<0 \\ \{\partial_z+(\mathrm{i}f_\epsilon-F\partial_x)^2\}\,G= \\ \quad -\delta(x-\widetilde{x})\,\delta(y-\widetilde{y}) \text{ at } z=0\end{array}\right\} \text{ if } \widetilde{z}=0 \tag{2}$$

The sign difference between the term $+F\,\partial/\partial x$ that appears in the free-surface boundary condition (1b) satisfied by the flow potential $\phi(\mathbf{x})$ and the term $-F\,\partial/\partial x$ that appears in the free-surface boundary conditions satisfied by the Green function $G(\mathbf{x}, \widetilde{\mathbf{x}})$ stems from the differentiation with respect to the coordinates of the source point $\mathbf{x}$ (rather than differentiation with respect to the coordinates of the flow field point $\widetilde{\mathbf{x}}$) that is used in (2). Indeed, the Green function $G(\mathbf{x}, \widetilde{\mathbf{x}})$ defined by (2) is a function of $x-\widetilde{x}$, and the term $-F\,\partial/\partial x$ in the free-surface boundary conditions in (2) yields $+F\,\partial/\partial\widetilde{x}$ as in the free-surface boundary condition (1b). Thus, the Green function $G(\mathbf{x}, \widetilde{\mathbf{x}})$ defined by (2) represents the velocity potential of the flow created at a point $\widetilde{\mathbf{x}} \equiv (\widetilde{x}, \widetilde{y}, \widetilde{z} \le 0)$ by a unit source located at a point $\mathbf{x} \equiv (x, y, z < 0)$ or by a unit flux at a point $\mathbf{x} \equiv (x, y, z = 0)$ of the free surface.

This Green function can be expressed as

$$4\pi G = G^S + G^F/\pi \tag{3a}$$

where G^S is defined in terms of elementary free-space singularities (Rankine sources), and G^F accounts for free-surface effects and is given by a double Fourier integral. Specifically, G^S is defined as

$$G^S \equiv \frac{-1}{r} + \frac{1}{r'} \text{ where } \left\{\begin{array}{l} r \equiv \sqrt{(\widetilde{x}-x)^2+(\widetilde{y}-y)^2+(\widetilde{z}-z)^2} \\ r' \equiv \sqrt{(\widetilde{x}-x)^2+(\widetilde{y}-y)^2+(\widetilde{z}+z)^2}\end{array}\right\} \tag{3b}$$

The free-surface component G^F in (3) is given by a Fourier superposition of elementary waves $\widetilde{\mathcal{E}}$ and $\mathcal{E}$:

$$G^F \equiv \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha\, \frac{\widetilde{\mathcal{E}}\,\mathcal{E}}{\Delta + \mathrm{i}\epsilon\,\Delta_f} \text{ where } \left\{\begin{array}{c}\widetilde{\mathcal{E}} \equiv e^{k\widetilde{z}+\mathrm{i}(\alpha\widetilde{x}+\beta\widetilde{y})} \\ \mathcal{E} \equiv e^{kz-\mathrm{i}(\alpha x+\beta y)} \\ 0<\epsilon\ll 1\end{array}\right\} \text{ and } \left\{\begin{array}{c} k \equiv \sqrt{\alpha^2+\beta^2} \\ \Delta \equiv (f+F\alpha)^2-k \\ \Delta_f \equiv 2\,(f+F\alpha)\end{array}\right\} \tag{3c}$$

Here, k is the wavenumber, the function $\Delta(\alpha, \beta; f, F)$ is related to the dispersion relation $\Delta = 0$, and $\Delta_f \equiv \partial\Delta/\partial f$ denotes the derivative of the dispersion function Δ with respect to the wave frequency f.

The Fourier representation (3c) contains waves as well as a nonoscillatory local flow component. Global analytical approximations, valid within the entire flow region $\widetilde{z} + z \le 0$, to the local flow component L and its gradient ∇L are given in [2] for the special case $F = 0$ and in [3,4] for the special case $f = 0$, and the Green functions for these two special cases can easily be evaluated. No such analytical approximation exists for the Green function defined by (3c) in the general case $\tau \ne 0$.

4. Classical boundary-integral relations

Application of Green's classical identity to the flow potential $\phi(\mathbf{x})$ and the Green function $G(\mathbf{x}, \widetilde{\mathbf{x}})$ in the mean flow region D bounded by the free surface Σ_F and the ship hull surface Σ_H yields

$$\int_D dv\,\phi\,\nabla^2 G = \int_{\Sigma_F} dx\,dy\,(\phi\,G_z - G\,\phi_z) + \int_{\Sigma_H} da\,(G\,q_{_H} - \phi\,G_n) \tag{4}$$

where the Laplace equation (1a) and the boundary condition (1c) at the ship hull surface Σ_H were used, dv denotes the differential element of volume of D, and $G_n \equiv \nabla G \cdot \mathbf{n}$ is the derivative of G along the unit vector $\mathbf{n}$ normal to Σ_H. As was already noted, $\mathbf{n}$ points into the water.

The integrand of the integral over the free surface Σ_F can be expressed as

$$\phi\,G_z - G\,\phi_z = \phi\,[\partial_z + (\mathrm{i}f_\epsilon - F\partial_x)^2]\,G - G\,[\partial_z + (\mathrm{i}f_\epsilon + F\partial_x)^2]\,\phi + \partial_x\,[2\mathrm{i}f_\epsilon F\,G\,\phi + F^2(G\,\phi_x - \phi\,G_x)]$$

This relation and the Green identity (4) then yield

$$\int_D dv\,\phi\,\nabla^2 G - \int_{\Sigma_F} dx\,dy\,\phi\,[\partial_z + (\mathrm{i}f_\epsilon - F\partial_x)^2]\,G = \int_\Gamma dy\,[2\mathrm{i}f_\epsilon F\,G\,\phi + F^2(G\,\phi_x - \phi\,G_x)]$$
$$+ \int_{\Sigma_H} da\,(G\,q_{_H} - \phi\,G_n) - \int_{\Sigma_F} dx\,dy\,G\,\pi^F$$

where π^F is given by (1b) and Stokes theorem was used to transform a surface integral over the undisturbed free surface Σ_F into a line integral around the mean ship waterline Γ. Equations (2) then yield

$$\widetilde{C}\,\widetilde{\phi} = \int_\Gamma dy\,[2\mathrm{i}f_\epsilon FG\phi + F^2(G\phi_x - \phi G_x)] + \int_{\Sigma_H} da\,(Gq_{_H} - \phi G_n) - \int_{\Sigma_F} dx\,dy\,G\pi^F \tag{5}$$

$$\text{where } \widetilde{C} \equiv \int_D dv\,\nabla^2 G - \int_{\Sigma_F} dx\,dy\,[\partial_z + (\mathrm{i}f_\epsilon - F\partial_x)^2]\,G \tag{6}$$

The relations (2) show that one has $\widetilde{C}=1$ for points $\widetilde{\mathbf{x}} \equiv (\widetilde{x},\widetilde{y},\widetilde{z})$ located inside the flow region $D\cup\Sigma_F$, $\widetilde{C}=0$ for points $\widetilde{\mathbf{x}}$ outside the flow region, i.e. within the region $D'\cup\Sigma'_F$ inside the mean wetted hull surface Σ_H, and $\widetilde{C}=1/2$ for points $\widetilde{\mathbf{x}}$ of the hull surface $\Sigma_H\cup\Gamma$. This classical result holds for $\widetilde{z}\le 0$.

5. Modified boundary-integral relation

$$\text{Define } \widetilde{C}' \equiv \int_{D'} dv\,\nabla^2 G - \int_{\Sigma'_F} dx\,dy\,[\partial_z + (\mathrm{i}f_\epsilon - F\partial_x)^2]\,G \tag{7a}$$

The relations (2), (6) and (7a) show that one has $\widetilde{C}+\widetilde{C}'=1$ for all points $\widetilde{\mathbf{x}}$ in the lower half space $\widetilde{z}\le 0$. Successive applications of the divergence theorem for the region D' and Stokes' theorem for the undisturbed waterplane Σ' located inside the mean waterline Γ yield

$$\widetilde{C}' = \int_{\Sigma_H} da\,G_n - \int_{\Sigma'_F} dx\,dy\,(\mathrm{i}f_\epsilon - F\partial_x)^2 G = \int_{\Sigma_H} da\,G_n - \int_\Gamma dy\,(f_\epsilon^2 G^x + 2\mathrm{i}f_\epsilon FG - F^2 G_x) \tag{7b}$$

Addition of the term $\widetilde{C}'\widetilde{\phi}$ on both sides of the classical boundary-integral relations (5), with $\widetilde{C}'$ given by (7a) or (7b) on the left or right sides of (5), yields the modified boundary-integral relation

$$\widetilde{\Gamma}\widetilde{\phi} = \int_{\Sigma_H} da\,[Gq_{_H} - (\phi-\widetilde{\phi})G_n] + \int_\Gamma dy\,[(2\mathrm{i}\tau G - F^2 G_x)(\phi-\widetilde{\phi}) + F^2 G\phi_x] - \int_{\Sigma_F} dx dy\,G\pi^F \tag{8a}$$

$$\text{or } \widetilde{\phi} = \int_{\Sigma_H} da\,[\widetilde{G}q_{_H} - (\phi-\widetilde{\phi})\widetilde{G}_n] + \int_\Gamma dy\,[(2\mathrm{i}\tau\widetilde{G} - F^2\widetilde{G}_x)(\phi-\widetilde{\phi}) + F^2\widetilde{G}\phi_x] - \int_{\Sigma_F} dx dy\,\widetilde{G}\pi^F \tag{8b}$$

$$\text{where } \widetilde{\Gamma} \equiv 1 + f^2\!\int_\Gamma dy\,G^x = 1 - f^2\!\int_{\Sigma'_F} dx dy\,G \text{ and } \widetilde{G}(\mathbf{x},\widetilde{\mathbf{x}}) \equiv G(\mathbf{x},\widetilde{\mathbf{x}})/\widetilde{\Gamma}(\widetilde{\mathbf{x}}) \tag{8c}$$

denotes a modified Green function [5]. The boundary-integral relations (8a) and (8b) hold for all flow-field points $(\widetilde{x},\widetilde{y},\widetilde{x}\le 0)$ and are equivalent to the three classical relations (5) with $\widetilde{C}=1$, 0 or 1/2 for points $\widetilde{\mathbf{x}}$ inside the flow region $D\cup\Sigma_F$, within the region $D'\cup\Sigma'_F$ or at the boundary $\Sigma_H\cup\Gamma$. In the special case $F=0$, i.e. for wave diffraction-radiation without forward speed, the boundary-integral relations (8) are identical to the relation given in [5]. In the special case $f=0$, i.e. for steady flow around a ship advancing in calm water, the hull flux $q_{_H}$ is given by $q_{_H}=n^x$ and equations (8) yield

$$\widetilde{\phi} = \int_{\Sigma_H} da\,[Gn^x - (\phi-\widetilde{\phi})G_n] - F^2\!\int_\Gamma dy\,[G_x(\phi-\widetilde{\phi}) - G\phi_x] - \int_{\Sigma_F} dx\,dy\,G\pi^F \tag{9}$$

The thin band of water located between the plane $z=0$ and the linear approximation $z=F^2(\phi_x-p^F)$ to the free-surface elevation yields a linear contribution to the term Gn^x in the integrand of the integral over the hull surface Σ_H in (9) that exactly cancels out the term $G\phi_x$ in the integral around the ship waterline Γ, as is shown in [1] and is easily verified. Within the framework of this consistent linear flow model of steady ship waves, called Neumann-Michell theory, the relation (9) then becomes

$$\widetilde{\phi} = \int_{\Sigma_H} da\,[Gn^x - (\phi-\widetilde{\phi})G_n] - F^2\!\int_\Gamma dy\,G_x(\phi-\widetilde{\phi})\,dy + F^2\!\int_{\Sigma_F} dx\,dy\,G_x p^F \tag{10}$$

Here, the free-surface flux q^F in (1b) is assumed to be nil, and Stokes' theorem was applied to express the integral of $(Gp^F)_x$ over the free surface Σ_F as a line integral around the ship waterline Γ.

6. Embedding of the waterline integral into the hull-surface integral

The Rankine component G^S in the basic decomposition (3a) of the Green function G is nil at the free-surface plane $z=0$. The component G^S, and its derivative G_x^S therefore do not appear in the integrals over the free surface Σ_F and the waterline Γ in the boundary-integral relations (8) and (10). Expression (3c) shows that the contribution of the free-surface component G^F in (3a) to the integrals over the hull surface Σ_H and the waterline Γ on the right side of (8) is given by

$$\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha\,\frac{\widetilde{\mathcal{E}}(A^H + A^\Gamma)}{\Delta + \mathrm{i}\epsilon\,\Delta_f} \text{ where } \left\{ \begin{array}{l} A^H \equiv \int_{\Sigma_H} da\,[q_{_H} + \mathrm{i}(\alpha n^x + \beta n^y + \mathrm{i}kn^z)(\phi-\widetilde{\phi})]\,\mathcal{E} \\ A^\Gamma \equiv \int_\Gamma dy\,[\mathrm{i}(2\tau + F^2\alpha)(\phi-\widetilde{\phi}) + F^2\phi_x]\,\mathcal{E} \end{array} \right\} \tag{11}$$

The line integral around the waterline Γ that defines the wave-amplitude function A^Γ in the Fourier-Kochin representation (11) is now embedded into the surface integral over Σ_H.

At a point $\mathbf{x}=(x,y,0)$ of the mean waterline Γ, the unit vector $\mathbf{n}\equiv(n^x,n^y,n^z)$ normal to the ship hull surface Σ_H, the unit vector $\mathbf{t}\equiv(n^y,-n^x,0)/1^z$ where $1^z\equiv\sqrt{1-(n^z)^2}$ tangent to Γ, and the unit vector $\mathbf{s}\equiv\mathbf{n}\times\mathbf{t}\equiv(n^z n^x/1^z, n^z n^y/1^z, -1^z)$ tangent to Σ_H form a local system of three orthogonal unit vectors. The vector $\mathbf{s}$ is nearly vertical and points downward. The point $\mathbf{x}' \equiv \mathbf{x} - z\,\mathbf{s}(\mathbf{x})$, where $\mathbf{x} \equiv (x, y, 0)$ is a point of the ship waterline Γ, $z \le 0$ and $\mathbf{s}(\mathbf{x})$ is the unit vector $\mathbf{s}$ at the point $\mathbf{x}$, is given by $\mathbf{x}' \equiv (x', y', z') = (x - z n^z n^x/1^z, y - z n^z n^y/1^z, z 1^z)$. The point $\mathbf{x}'$ nearly rests on the hull surface Σ_H over the upper part of the port and starboard sides of a common ship hull, specifically for $-\delta \le z \le 0$ where δ is smaller than the local draft $d \equiv D/L$ of the ship hull. Integration of the function $(1 + z/\delta)^2 \mathcal{E}$, where $\mathcal{E}$ is the elementary wave function defined by (3c), along the line $\mathbf{x} - z\,\mathbf{s}(\mathbf{x})$ yields

$$\int_{-\delta}^{0} dz (1 + z/\delta)^2 e^{k z' - \mathrm{i}(\alpha x' + \beta y')} = \frac{\delta/3}{K^\delta} e^{-\mathrm{i}(\alpha x + \beta y)} \text{ where} \tag{12}$$

$$K^\delta \equiv (k_\delta^3/6)/(1 - k_\delta + k_\delta^2/2 - e^{-k_\delta}) \text{ and } k_\delta \equiv [k\sqrt{1-(n^z)^2} + \mathrm{i} n^z(\alpha n^x + \beta n^y)/\sqrt{1-(n^z)^2}\,]\,\delta \tag{13}$$

One has $K^\delta \sim 1$ as $k_\delta \to 0$ and $3K^\delta/\delta \sim 1$ as $k_\delta \to \infty$. The relation (12) can be used to express the waterline integral in (11) as the hull-surface integral

$$A^\Gamma = -\int_{\Sigma_H} da \frac{K^\delta}{\delta/3} H^\delta \Big(1 + \frac{z}{\delta}\Big)^2 [\mathrm{i}(2\tau + F^2\alpha)(\phi^\Gamma - \widetilde{\phi}) + F^2 \phi_x^\Gamma] \frac{n^x}{1^z} \mathcal{E} \text{ where } H^\delta \equiv H(\zeta + \delta) \tag{14}$$

$H(\cdot)$ in (14) is the Heaviside unit-step function, and ϕ^Γ and ϕ_x^Γ mean that ϕ and ϕ_x are evaluated at the mean waterline Γ. The representation (14) smoothly spreads the waterline integral A^Γ in (11) over the upper part $-\delta \le z \le 0$ of the ship hull surface Σ_H.

The hull-surface integral A^H in (11) can be expressed as

$$A^H = \int_{\Sigma_H} da\, [q_H^* + \mathrm{i}(\alpha n^x + \beta n^y + \mathrm{i} k n^z)(\phi^* - \widetilde{\phi}) + H^\delta \Big(1 + \frac{z}{\delta}\Big)^2 \{q_H^\Gamma + \mathrm{i}(\alpha n^x + \beta n^y + \mathrm{i} k n^z)(\phi^\Gamma - \widetilde{\phi})\}]\,\mathcal{E} \tag{15}$$

$$\text{where } \begin{Bmatrix} q_H^* \\ \phi^* - \widetilde{\phi} \end{Bmatrix} \equiv (1 - H^\delta) \begin{Bmatrix} q_H \\ \phi - \widetilde{\phi} \end{Bmatrix} + H^\delta \left[\begin{Bmatrix} q_H \\ \phi - \widetilde{\phi} \end{Bmatrix} - \Big(1 + \frac{z}{\delta}\Big)^2 \begin{Bmatrix} q_H^\Gamma \\ \phi^\Gamma - \widetilde{\phi} \end{Bmatrix} \right] \tag{16a}$$

Expressions (15) and (14) show that (11) can finally be expressed as

$$\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \, \frac{\widetilde{\mathcal{E}} A}{\Delta + \mathrm{i}\epsilon \Delta_f} \text{ where } A = \int_{\Sigma_H} da\, [q_H^* + \mathrm{i}(\alpha n^x + \beta n^y + \mathrm{i} k n^z)(\phi^* - \widetilde{\phi}) + H^\delta \Big(1 + \frac{z}{\delta}\Big)^2 a_H^\Gamma]\,\mathcal{E} \tag{16b}$$

$$\text{and } a_H^\Gamma \equiv q_H^\Gamma + \mathrm{i}\left[\left(\alpha - (2\tau + F^2\alpha)\frac{3K^\delta/\delta}{\sqrt{1-(n^z)^2}}\right) n^x + \beta n^y + \mathrm{i} k n^z\right](\phi^\Gamma - \widetilde{\phi}) - \frac{(3K^\delta/\delta)\, n^x}{\sqrt{1-(n^z)^2}} F^2 \phi_x^\Gamma \tag{16c}$$

In (16b), q_H^* and ϕ^* are given by (16a); and K^δ in (16c) is given by (13). The terms multiplied by $H^\delta \equiv H(\zeta + \delta)$ in (16a) and (16b) are nil for $z \le -\delta$, where $0 < \delta < d$ is the local draft of the ship hull. The second term in (16a) vanishes at the mean waterline Γ, where $z = 0$. Expression (16a) yields smooth variations of q_H^* and ϕ^* at the transition $z = -\delta$. In the special case $f = 0$, (16c) becomes

$$a_H^\Gamma \equiv n^x + \mathrm{i}\left[\left(1 - \frac{F^2\, 3K^\delta/\delta}{\sqrt{1-(n^z)^2}}\right)\alpha n^x + \beta n^y + \mathrm{i} k n^z\right](\phi^\Gamma - \widetilde{\phi}) \tag{17}$$

7. Conclusions

The boundary-integral representation (8), and the related Fourier-Kochin representation (16) of the wave-component associated with the integrals over the ship hull surface Σ_H and the ship waterline Γ in (8), is significantly better suited for accurate numerical evaluation than the classical formulation (5). In particular, (8) and (16) involve $\phi - \widetilde{\phi}$ instead of ϕ in (5). Moreover, (8) and (16b) do not involve a line integral around the ship waterline Γ. Thus, numerical cancellations that may occur between the hull-surface and waterline integrals (which are difficult to evaluate with high accuracy) in the classical boundary-integral formulation (5) occur among analytical functions (that can be evaluated very accurately) in the function a_H^Γ defined by (16c).

In the special case $f = 0$, i.e. for ship waves in calm water, the boundary-integral relation (10) and the related expressions (16a), (16b) and (17) provide an interesting alternative to the formulation of the NM theory [1]. This theory has been largely validated and applied for hull-form optimization.

The term $F^2 G \phi_x$ in (8a) is eliminated in (10) via the consideration of a consistent linear flow model. However, the formulation of a consistent linear flow model is far less clear if $f \neq 0$.

Other issues, notably the filtering of short waves, may have important effects on numerical solutions.

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