

# Transient wave run-up on cylinders due to wavefronts

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Further to the work by Chen *et al.* [1] on waves generated by an impulsive perturbation around an infinite vertical circular cylinder, we apply the Fourier-Laguerre spectral method to solve the wave diffraction problem by a vertical cylinder in the time domain. For the choice of incoming waves, transient waves with wavefront is adopted in present work, which is much more general than steady-state plane progressive waves. While few analysis related to wavefront is made due to its complexity in numerical computations. The present solution is verified by comparing with analytical transient diffraction solutions.

## 1 Fourier-Laguerre spectral method

A Cartesian coordinate system  $Oxyz$  is defined with the  $Oxy$  plane on the undisturbed free surface and  $Oz$  axis pointing positively upward. Consider an infinitely long vertical cylinder fixed in deepwater with its axis coinciding with the  $Oz$  axis. Applying the Green's theorem in the fluid domain, we can write the velocity potential as:

$$\Phi(P, t) = \int_0^t \iint_C [\Phi_n(Q, \tau)G(P, t, Q, \tau) - \Phi(Q, \tau)G_n(P, t, Q, \tau)] dSd\tau \quad (1)$$

where  $P = (x, y, z)$  and  $Q = (\xi, \eta, \zeta)$  represent flow-field point and source point, respectively;  $C$  denotes the cylinder surface; and  $G$  is the transient Green function defined as [2]:

$$4\pi G(P, t, Q, \tau) = \delta(t - \tau)G^0 + H(t - \tau)G^f \quad (2)$$

In (2),  $\delta(\cdot)$  and  $H(\cdot)$  are delta function and Heaviside step function;  $G^0$  and  $G^f$  are defined by:

$$G^0 = -\frac{1}{\sqrt{R^2 + (z - \zeta)^2}} + \frac{1}{\sqrt{R^2 + (z + \zeta)^2}} \quad \text{and} \quad G^f = -2 \int_0^\infty e^{k(z+\zeta)} J_0(kR) \sqrt{gk} \sin[\sqrt{gk}(t - \tau)] dk \quad (3)$$

where  $R$  is defined as  $R = \sqrt{(x - \xi)^2 + (y - \eta)^2}$ , and  $J_0(\cdot)$  denotes the zeroth-order Bessel function of the first kind. On the cylindrical surface with a unit radius, the velocity potential  $\Phi$  and its normal derivative  $\Phi_n$  are expanded into the Fourier-Laguerre series:

$$\Phi = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \phi_{mn}(\tau) \mathcal{L}_m(-\zeta) e^{in\varphi} \quad \text{and} \quad \Psi = \Phi_n = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \psi_{mn}(\tau) \mathcal{L}_m(-\zeta) e^{in\varphi} \quad (4)$$

where the  $m$ -order Laguerre function  $\mathcal{L}_m(v)$  is defined in [3]. By constructing the boundary integral equation (1) on the cylinder surface in the sense of Galerkin collocation via integrating a test function  $\mathcal{L}_j(-z)e^{-i\ell\varphi}$  over the cylinder surface, we have the following expression [1]:

$$\phi_{j\ell}(t) + \sum_{m=0}^M \phi_{m\ell}(t) \mathcal{H}_{j\ell, m\ell}^0 = \sum_{m=0}^M \psi_{m\ell}(t) \mathcal{G}_{j\ell, m\ell}^0 + \sum_{m=0}^M \int_0^t \{ \psi_{m\ell}(\tau) \mathcal{G}_{j\ell, m\ell}^C - \phi_{m\ell}(\tau) \mathcal{H}_{j\ell, m\ell}^C \} d\tau \quad (5)$$

where  $\mathcal{G}_{j\ell, m\ell}^0$ ,  $\mathcal{H}_{j\ell, m\ell}^0$ ,  $\mathcal{G}_{j\ell, m\ell}^C$ ,  $\mathcal{H}_{j\ell, m\ell}^C$  are five-fold integrals. By applying the orthogonal properties of the Fourier series and Laguerre functions, they can be simplified to single integrals:

$$\{ \mathcal{G}_{j\ell, m\ell}^0, \mathcal{H}_{j\ell, m\ell}^0 \} = 4 \int_0^\infty \left[ \frac{(2k-1)^{|m-j|-1}}{(2k+1)^{|m-j|+1}} (1 - 2k\delta_{mj}) + \frac{(2k-1)^{m+j}}{(2k+1)^{m+j+2}} \right] J_\ell(k) \{ J_\ell(k), kJ'_\ell(k) \} dk \quad (6)$$

$$\{ \mathcal{G}_{j\ell, m\ell}^C, \mathcal{H}_{j\ell, m\ell}^C \} = -8 \int_0^\infty \frac{(2k-1)^{m+j}}{(2k+1)^{m+j+2}} \sqrt{gk} J_\ell(k) \{ J_\ell(k), kJ'_\ell(k) \} \sin[(t - \tau) \sqrt{gk}] dk \quad (7)$$

The single integrals in (6) are independent of time. The single integrals in (7) are time-dependent and very oscillatory because of products of dual Bessel functions and trigonometric function. Integrating along the real axis is nearly impossible when the time parameter is large. It shall be integrated along an appropriate path in the complex plane. The integrating path, on which the integrand is exponentially-decreasing, can be determined approximately by analysing the phase function related to the time parameter in the complex plane and the integral evaluation can be efficient and accurate.

## 2 Transient diffraction wave run-up and wave loads

The transient incoming waves with wavefront are generated by a harmonically oscillating flexible plate, and then the velocity potential  $\Phi^I(\rho, \theta, z, t)$  with respect to the reference of cylinder is written as [4]:

$$\Phi^I = \frac{Ag}{\omega} e^{k_0 z} \sin(k_0 \rho \cos \theta + k_0 L) \sin(\omega t) + \frac{2Agk_0}{\omega \pi} \int_0^\infty e^{kz} \cos(k\rho \cos \theta + kL) \frac{\omega}{k^2 - k_0^2} \frac{\sin(\beta t)}{\beta} dk \quad (8)$$

with  $\beta = \sqrt{gk}$ . The parameter  $L$  is the distance between the wavemaker and the cylinder centre.

According to the body boundary condition, the Fourier-Laguerre coefficients  $\psi_{mn}$  associated with normal derivative of diffraction potential is written as:

$$\begin{aligned} \psi_{mn}(t) &= \frac{1}{2\pi} \int_{-\infty}^0 \int_{-\pi}^{\pi} \Phi_n^D \mathcal{L}_m(-z) e^{-in\varphi} d\varphi dz \\ &= -\frac{Agk_0}{\omega} \frac{(k-1/2)^m}{(k+1/2)^{m+1}} \sin(k_0 L + n\pi/2) J'_n(k_0) \sin(\omega t) \\ &\quad - \frac{2Agk_0}{\pi} \int_0^\infty \frac{(k-1/2)^m}{(k+1/2)^{m+1}} \frac{k}{k^2 - k_0^2} \cos(kL + n\pi/2) J'_n(k) \frac{\sin(\beta t)}{\beta} dk \end{aligned} \quad (9)$$

The Fourier-Laguerre coefficients  $\phi_{mn}(t)$  are obtained by solving the linear equation system (5). Then, the velocity potential distribution over the cylinder surface is obtained by using (4). Therefore, we can get the wave run-up on the cylinder and the wave excitation force acting on the cylinder.

The wave run-up on the cylinder  $\eta$  non-dimensionalized by the wave amplitude  $A$  consists of incoming waves  $\eta^I$  and diffracted waves  $\eta^D$ :

$$\eta = \eta^I + \eta^D \quad \text{with} \quad \{\eta^I, \eta^D\} = -\frac{1}{Ag} \{\Phi_t^I, \Phi_t^D\} \Big|_{z=0} \quad (10)$$

where  $\eta^I$  and  $\eta^D$  are given by:

$$\eta^I = -\sin(k_0 \cos \theta + k_0 L) \cos(\omega t) - \frac{2k_0}{\pi} \int_0^\infty \cos(k \cos \theta + kL) \frac{1}{k^2 - k_0^2} \cos(\beta t) dk \quad (11a)$$

and

$$\eta^D = -\frac{1}{Ag} \frac{\partial}{\partial t} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \phi_{mn}(t) e^{in\varphi} = -\frac{1}{Ag} \frac{\partial}{\partial t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_n \phi_{mn}(t) \cos(n\varphi) \quad (11b)$$

with  $\epsilon_0 = 1$  and  $\epsilon_n = 2$  for  $n > 0$ .

For the wave excitation force exerting on the cylinder, the non-dimensional form  $F_x$  with respect to  $\rho g A$  is expressed as:

$$F_x = F_x^I + F_x^D \quad \text{with} \quad \{F_x^I, F_x^D\} = \frac{1}{gA} \int_{-\infty}^0 \int_{-\pi}^{\pi} \{\Phi_t^I, \Phi_t^D\} \cos \varphi d\varphi dz \quad (12)$$

where  $F_x^I$  and  $F_x^D$  are given by:

$$F_x^I = \frac{2\pi}{k_0} J_1(k_0) \cos(k_0 L) \cos(\omega t) - 4k_0 \int_0^\infty \frac{J_1(k) \sin(kL) \cos(\beta t)}{k(k^2 - k_0^2)} dk \quad (13a)$$

and

$$F_x^D = \frac{4\pi}{gA} \frac{\partial}{\partial t} \sum_{m=0}^{\infty} (-1)^m \phi_{m1}(t) \quad (13b)$$

from which it can be seen that only  $\phi_{m1}$  makes contribution to the diffraction wave load  $F_x^D$ .

### 3 Analytical formulations

Analytical diffraction potential  $\Phi^D(P, t)$  at a flow-field point  $P$  and time  $t$  can be decomposed into instantaneous and memorial terms:

$$\Phi^D(P, t) = \iint_C \Psi(P, Q)v(Q, t)dS + \int_0^t d\tau \iint_C v(Q, \tau)\chi(P, Q, t - \tau)dS \quad (14)$$

where  $v(Q, t) = -\partial\Phi^I/\partial r$  is the normal velocity on the cylinder surface associated with incoming waves. The fundamental solutions  $\Psi$  and  $\chi$  are respectively instantaneous and memorial components of the potential due to an impulsive source distribution on the cylinder surface. More details for the analytical solutions of  $\Psi$  and  $\chi$  in infinite waterdepth and corresponding diffraction force formulations can be found in [5]. The wave run-up  $\eta^D$  contributed from diffraction potential  $\Phi^D$  is expressed as:

$$\begin{aligned} \eta^D = & -\sum_{n=0}^{\infty} \epsilon_n \cos(n\theta) \sin(k_0 L + n\pi/2) J'_n(k_0) P_n(k_0) \\ & - \frac{2k_0}{\pi} \sum_{n=0}^{\infty} \epsilon_n \cos(n\theta) \int_0^{\infty} \cos(kL + n\pi/2) J'_n(k) \frac{1}{k^2 - k_0^2} P_n(k) dk \end{aligned} \quad (15)$$

where  $P_n(k)$  and  $F_n(k)$  are respectively defined as:

$$P_n(k) = \cos(\beta t) \left[ -\text{Re}\{F_n(k)\} + \frac{2k}{\pi} \int_0^{\infty} \frac{\text{Im}\{F_n(\hat{k})\}}{\hat{k}^2 - k^2} d\hat{k} \right] + \frac{2k}{\pi} \int_0^{\infty} \frac{\cos(\hat{\beta} t) - \cos(\beta t)}{\hat{k}^2 - k^2} \text{Im}\{F_n(\hat{k})\} d\hat{k} \quad (16a)$$

and

$$F_n(k) = \frac{2H_n^{(1)}(k)}{H_{n-1}^{(1)}(k) - H_{n+1}^{(1)}(k)} \quad (16b)$$

with  $H_n^{(1)}(\cdot)$  being the Hankel function of the first kind with order  $n$ .

### 4 Results and discussions

The transient waves diffracted by a vertical cylinder have been given in [6] by using the classical frequency-domain solution in [7] to substitute the propagating terms in the representation of transient incoming waves. This method was extended to get the expression of transient diffraction potential and to evaluate the diffraction loads.

Unlike that indirect time domain method, we use the Fourier-Laguerre spectral method to determine  $\phi_{mn}$  by inputting  $\psi_{mn}$  given by (9). In fact, only the terms for  $n = 1$  are necessary to evaluate the transient diffraction wave loads defined by (13b). The transient diffraction wave run up (11) and wave loads (13) are evaluated for the frequency  $\omega = \sqrt{g}$  and depicted on Figure 1 and Figure 2 respectively for the distance  $L = 4\pi$ . More comparisons among results by using the method in [6], results by analytical formulations and those by the Fourier-Laguerre spectral method will be presented in the workshop.

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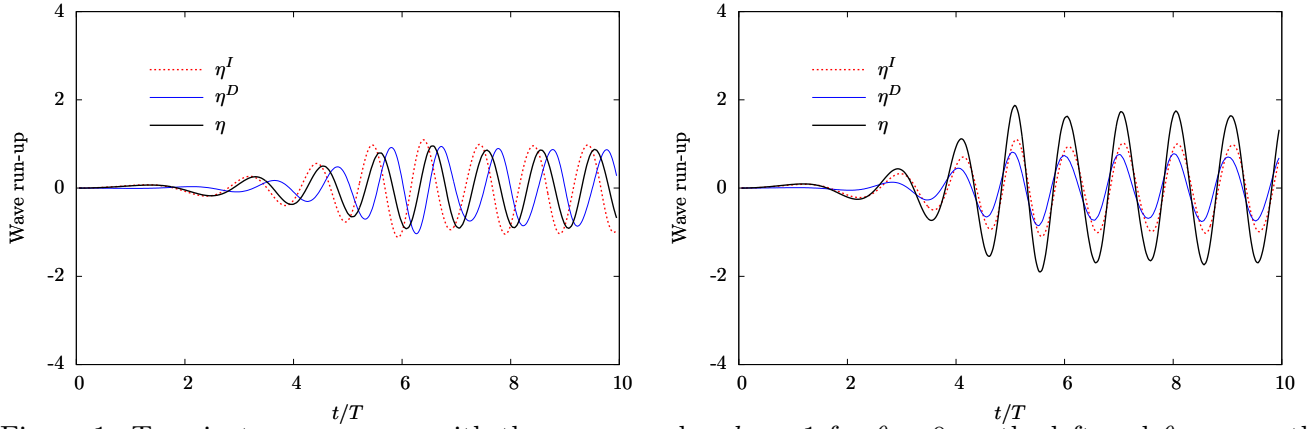


Figure 1: Transient wave run up with the wave number  $k_0 = 1$  for  $\theta = 0$  on the left and  $\theta = \pi$  on the right, incoming part  $\eta^I$  by dashed line, diffraction part  $\eta^D$  by thin solid line and resultant wave run up  $\eta^I + \eta^D$  by thick solid line.

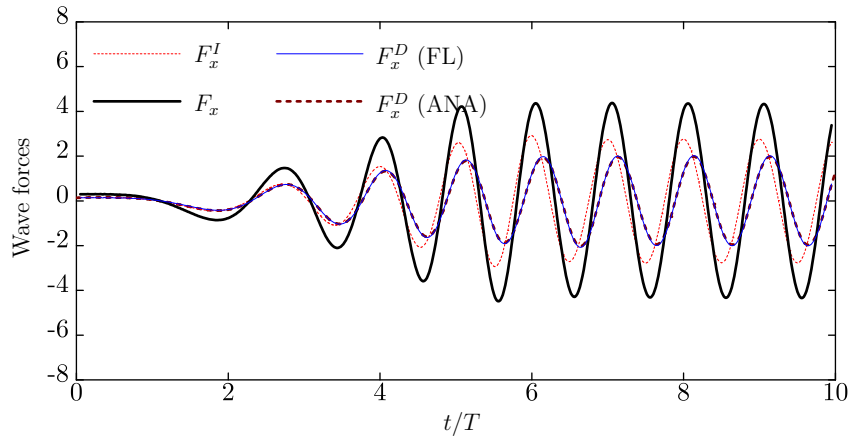


Figure 2: Transient wave loads for the distance  $L = 4\pi$  at the frequency  $\omega = \sqrt{g}$ , by thin dashed line for incoming wave part  $F_x^I$  (13a), by thin solid line for diffraction part  $F_x^D$  (13b), by thick solid line for the excitation load  $F_x$  (12) and by thick dashed line for the diffraction part obtained from analytical formulations.

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