Formation of waves riding on the forced free-surface flow

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Introduction A fully nonlinear problem on unsteady inviscid free surface flow generated by a totally submerged horizontal cylinder is considered semi-analytically. Our main purpose is to evaluate an impact of nonlinearity acting at the early stage of the flow when body starts moving near water surface. Here the role of nonlinearity is clarified by analysis of approximate solution which includes explicitly the higher-order nonlinear terms. It is shown that these terms can describe the formation of non-stationary small-scaled waves which propagate along the moving free surface with non-constant phase speed.

Boundary integral-differential formulation We consider a potential 2D flow of infinitely deep fluid caused by submerged circular cylinder of non-dimensional radius r, whose centre moves along the known trajectory $(x_c(t),\, y_c(t))$. Our analysis uses reduction of fully nonlinear Euler equations to the boundary integral-differential system of equations

$$\eta_t = v, \qquad u_t + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{u^2 - 2\eta_x uv - v^2}{1+\eta_x^2}\right) + \lambda\eta_x = 0, \tag{1}$$

$$v(x) + \int\limits_{-\infty}^{+\infty} (A(x,s) + r^2 A_r(x,s))v(s)ds = \int\limits_{-\infty}^{+\infty} (B(x,s) + r^2 B_r(x,s))u(s)ds + r^2 v_d(x) \tag{2}$$

for the wave elevation η together with normal component $v(x,t) = V - \eta_x U$ and tangential component $u(x,t) = U + \eta_x V$ of fluid velocity vector $(U(x,y,t), V(x,y,t))$ taken at the free surface $y = \eta(x,t)$ (see [1, 2, 3]). The real-valued operators A, B, A_r and B_r are given in complex form by the formulae

$$A(x,s) + iB(x,s) = \frac{iz'(x)}{\pi[z(x) - z(s)]}, \quad A_r(x,s) + iB_r(x,s) = \frac{iz'(x)}{\pi(\overline{z}(x) - \overline{c})^2 (z_*(x) - z(s))} \tag{3}$$

with $z(x) = x + i\eta(x)$ and $z(s) = s + i\eta(s)$. Here $z_* = c + r^2/(\overline{z} - \overline{c})$ is the inversion image of z with respect to the circle centered at the point $c = x_c + iy_c$ (bar denotes complex conjugate). Note that the kernels A and B correspond to the problem of free surface waves in deep fluid without cylinder whereas the terms A_r and B_r describe the interaction between the cylinder and free surface. The function v_d is normal velocity induced at the free surface by a dipole:

$$v_d(x) = \mathrm{Re}\, \frac{2ic'(t)z'(x)}{(z(x) - c)^2}. \tag{4}$$

The time variable t was omitted in (2) because it appears in this integral equation only as a parameter. It should be emphasised that the system (1),(2), being one-dimensional with respect to spatial variable x, arises from nonlinear water problem without any simplifying assumptions. Namely, differential equations (1) result here by excluding the pressure from boundary conditions at the free surface. In addition, integral equation (2) takes into account exact Neumann condition at the body surface which can be automatically satisfied by using the Milne-Thomson transformation. The parameter $\lambda = gh/u_0^2$ here is the squared inverse Froude number defined with characteristic speed of the cylinder u_0, initial submergence depth h, and gravity acceleration g.

Small-time asymptotic solution Small-time solution expansions were obtained starting from papers by Tyvand & Miloh [4, 5] devoted to the case of impulsive motion of circular cylinder. Similarly, we consider the unsteady flow which starts from rest $(u(x,0) = v(x,0) = \eta(x,0) = 0)$ and is caused by accelerated circular cylinder which trajectory is prescribed by

$$x_c(t) = \frac{t^2\cos\theta}{1+t^2}, \qquad y_c(t) = -1 + \frac{t^2\sin\theta}{1+t^2}$$

with constant angle θ assigning the direction of motion. This type of motion is characterised by a smooth start with a consequent gradual stop at an infinite time within the segment of unit length. We look for a solution in the form of power series with respect to the time variable t:

$$\eta(x,t) = t^2\eta_2(x) + t^3\eta_3(x) + \ldots, \quad u(x,t) = t^3 u_3(x) + t^4 u_4(x) + \ldots, \quad v(x,t) = tv_1(x) + t^2 v_2(x) + \ldots \tag{5}$$

The coefficients η_n, u_n and v_n are evaluated explicitly by recursive formulae which follow from the system of equations (1),(2). The key role is played by the integral equation (2) which provides calculating these coefficients by the perturbation procedure using small parameter r. Detailed description of this procedure can be found in our work [1]. Finally we obtain an explicit formula which gives the coefficients η_n for the free surface elevation η as follows

$$\eta_2(x) = \frac{1}{2}r^2(r^2 - 4)\Big(p'(x)\cos\theta - q'(x)\sin\theta\Big) + O(r^6), \quad \eta_3(x) = 0, \tag{6}$$

$$\eta_4(x) = r^2\Big(1-\frac{r^2}{4}\Big)\Big(2p'(x)\cos\theta - 2q'(x)\sin\theta + \Big(\cos 2\theta + \frac{\lambda}{6}\sin\theta\Big)p''(x) + q''(x)\Big(\frac{\lambda}{6}\cos\theta - \sin 2\theta\Big)\Big)+$$
$$+\frac{r^4}{9}\Big(p''''(x)\cos 2\theta - q''''(x)\sin 2\theta\Big) + \frac{r^4}{4}\Big(\Big(\sin 2\theta - \frac{\lambda}{3}\cos\theta\Big)p'(x) + q'(x)\Big(\frac{\lambda}{3}\sin\theta + \cos 2\theta\Big)\Big) + \frac{r^4}{3}\Big(p''(x) - \frac{7}{4}q'(x)\Big) + O(r^6)$$

where $p(x)$ and $q(x)$ are rational functions

$$p(x) = \frac{1}{1+x^2}, \qquad q(x) = \frac{x}{1+x^2}.$$

Neglecting here the terms of the order r^4 we obtain the known leading-order asymptotic solution

$$\eta(x,t) = 2r^2(t^2-t^4)\frac{2x\cos\theta + (1-x^2)\sin\theta}{(1+x^2)^2} + 2r^2t^4\Big[\frac{3x^2-1}{(1+x^2)^3}\Big(\cos 2\theta + \frac{\lambda}{6}\sin\theta\Big) + \Big(\frac{\lambda}{6}\cos\theta - \sin 2\theta\Big)\frac{x^3-3x}{(1+x^2)^3}\Big] + O(t^6+r^4). \tag{7}$$

Solution (7) was obtained originally in the paper [5] with use of bi-polar coordinates technique, and this leading order solution was also considered in paper [3] by the boundary equations technique. Note that the same solution (7) can be constructed immediately from the linearised version of integral-differential system of equations (1),(2). It is not surprising because we consider here non-stationary motion which starts smoothly from rest so that the leading-order solution is generated by the linear terms. Solution (6) is to take into account the higher-order terms describing nonlinear effects.

Vertical submersion of the cylinder In accordance with asymptotic solution (6) the flow regimes are determined by three parameters: radius of the cylinder r, direction of motion (angle θ) and acceleration rate (parameter λ). In the case of vertical submersion of the cylinder ($\theta = \pi/2$) the coefficients (6) determine the free surface elevation in the form

$$\eta(x,t) = t^2r^2(4-r^2)\frac{1-x^2}{2(1+x^2)^2} + t^4r^2\frac{\lambda+12-3(\lambda+6)x^2-6x^4}{3(1+x^2)^3} +$$
$$+ t^4r^4\frac{(\lambda+16)x^8 + (5\lambda+74)x^6 + 5(\lambda-18)x^4 - (\lambda-302)x^2 - 2\lambda - 62}{12(1+x^2)^5}. \tag{8}$$

This solution describes the formation of splash-jet over the vertically submerging cylinder which is shown in Fig. 1. The first term with t^2 in (8) dominates at the early stage when the cylinder starts moving from rest. Further, the term with t^4r^2 goes into action and makes free surface to reverse. Consequently, the vertical splash-jet starts forming when both these terms in (8) are balanced. At the same time, the term with t^4r^4 in (8) accounting for nonlinear effects starts working, so that the small-scale perturbations appear on the free surface. Thus nonlinear terms change not only the magnitude of the splash-jet together with its formation rate, but also cause slight wave-like perturbations on the free surface. The most interesting property of this flow regime is slight splitting of the top part of the vertical jet which can be observed in Fig. 1. This effect reminds the formation of small-amplitude waves during short-time period,

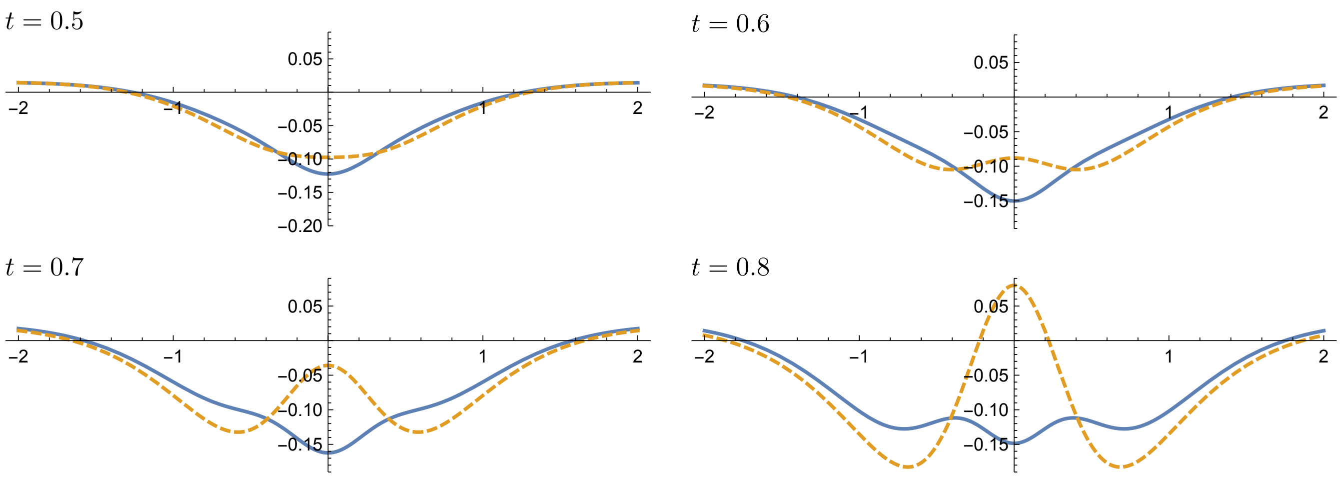

Fig 1: Deformation of the free surface above the submerging cylinder of radius $r = 0.6$ with $\lambda = 5$ predicted by the higher-order solution (6) (solid line) and by the leading-order solution (7) (dashed line).

when the buoyancy force and the inertia of the liquid mass are balanced at the leading order. Two waves appear symmetrically on the deflected free surface, and these waves start moving towards each other sliding down until they meet in the middle point above the cylinder forming the vertical splash-jet. Propagation speed and steepness of these waves depend essentially on the parameters λ and r.

The Figure 2a shows the tracks of the crests and depression points of riding waves during the time period when they exist. The Fig. 2b also captures the fragment of the flow domain which demonstrates the rotation of the free

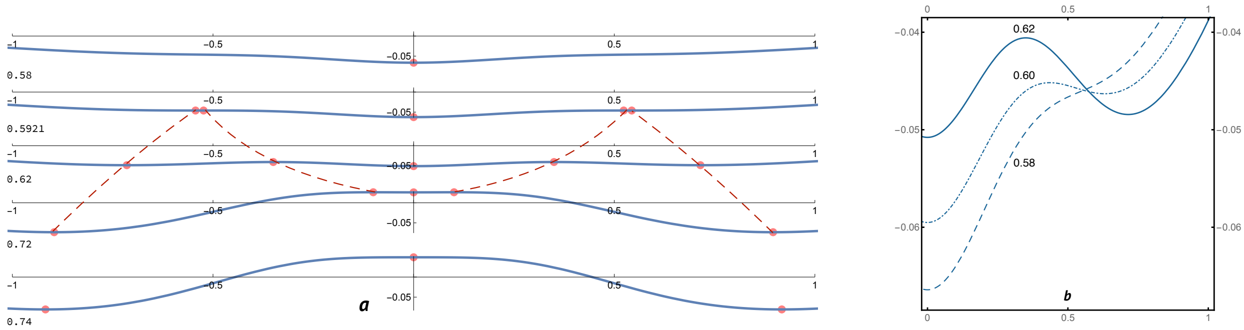

Fig 2: Formation of waves riding on the free surface caused by submerging cylinder of radius $r = 0.6$ with parameter $\lambda = 6$: (a) tracks of extremum points of the free surface during the time period [0.58, 0.74]; (b) fragments of rotating free surface at the time moments 0.58, 0.60, 0.62.

surface related to the axis which locates on the lee side of the wave. It could be supposed that formation of such small-amplitude waves, generated by the higher-order nonlinearity during the short-time equilibrium moment, causes development of instability which can probably lead to the breaking of a jet. It is interesting that direct numerical simulation of fully nonlinear water wave problem presented in [7] demonstrate similar qualitative behaviour of the free surface when the cylinder sinks freely (see Fig. 3, Fig. 4).

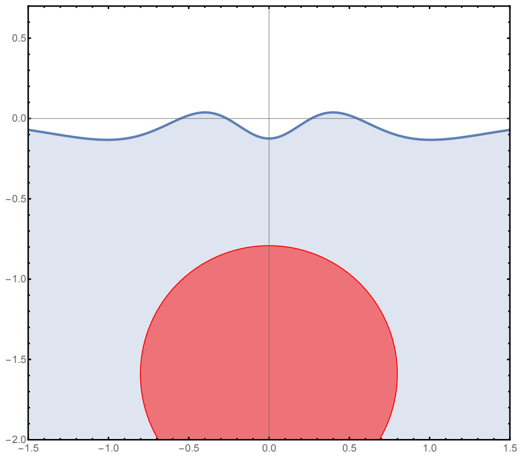

Fig 3: Deformation of the free surface for prescribed submersion of the cylinder with parameters $r = 0.8$, $\lambda = 6$ at time $t = 0.75$ predicted by the higher-order solution (8).

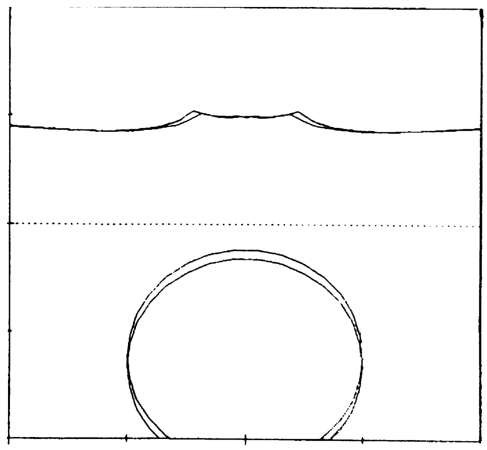

Fig 4: Free surface deformation for the freely sinking cylinder calculated numerically in [7] at dimensionless times $t = 4.833, 5.043$. Relation of the radius to initial submergence depth is $r/d = 0.8$.

Horizontal motion of the cylinder It is also interesting to consider in this context non-stationary formation of the waves generated by horizontal motion of the body. In this case ($\theta = 0$) the coefficients (6) determine the free surface elevation in the form

$$\eta(x,t) = t^2 r^2 (4 - r^2)\frac{x}{(1+x^2)^2} + t^4 r^2 \frac{(\lambda-12)x^3 + 18x^2 - 3(\lambda+4)x - 6}{3(x^2+1)^3} + \\ + t^4 r^4 \frac{4x^8 + (\lambda+12)x^7 + 14x^6 + (7\lambda+36)x^5 + 170x^4 + (11\lambda+36)x^3 - 362x^2 + (5\lambda+12)x + 26}{(1+x^2)^5}. \quad (9)$$

Haussling & Coleman [6] demonstrated numerically that nonlinear effects start to reveal themselves when the parameter r exceeds the value $r = 0.5$. In this case, calculation causes troubles at large time intervals due to numerical instabilities possibly related to the wave-breaking. Keeping this in mind, we compare the first-order solution (7) for $\theta = 0$ and the higher-order solution (6) with the well-known exact solution from linear theory. Namely, we refer here to the results obtained by Havelock [8] and Sretensky [9] who showed that deformation of the free surface can be determined explicitly via the integral formula

$$\eta(x,t) = 4r^2 \int_0^t \tau \int_0^\infty k e^{-k} \sin k(x - \tau^2) \cos\sqrt{\lambda k}(t-\tau)\, dk d\tau. \quad (10)$$

This relation presents explicitly the solution of linear theory for a circular cylinder at any time $t > 0$. Fig. 5 demonstrates the wave elevations over the cylinder predicted by the solution (10) in comparison with small-time asymptotic approximations (9) and (7). Strangely enough the higher-order solution (9) being essentially nonlinear is in better agreement with the linear solution (10) than the leading-order solution (7) also arising from the linear terms only. In other words nonlinear terms distinguishing the solutions (9) and (7) improve the last one noticeably. Indeed, as it can be seen from the Fig. 5, the higher-order solution (9) agrees with the linear solution (10) up to the time moment $t = 0.7$ whereas leading-order asymptotic solution starts to deviate at the time moment $t = 0.5$. Fig. 6 shows the flow

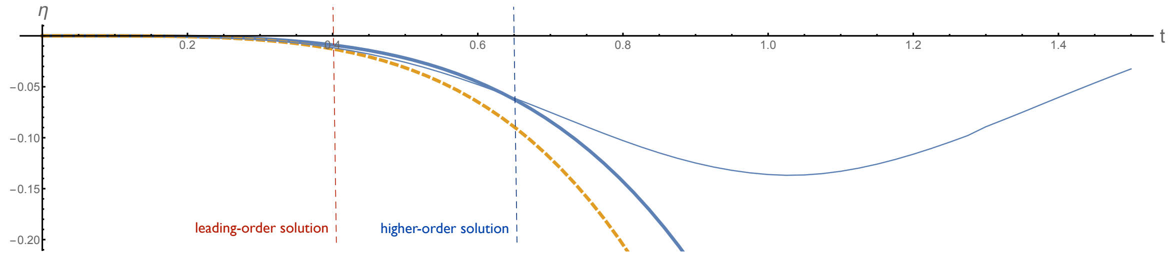

Fig 5: The free surface elevation in the point $x = 0$ caused by horizontal motion of the cylinder of radius $r = 0.5$ with parameter $\lambda = 5$ predicted by the higher order asymptotic solution (9) (solid line), by the lineading order asymptotic solution (7) (dashed line) and by the solution (10) of the linear theory (thin line).

domain when the cylinder starts its motion in horizontal direction. In the early development, when wave slopes are small, the three cases (7), (9), (10) are essentially identical. At later times visible differences naturally appear between first-order solution (7) and the higher-order solution (9). Namely, the leading-order small-time solution (7) overrates slightly the amplitude and disagrees totally with the phase of the wave generated by the body. At the same time, linear theory (10) and the higher-order asymptotic solution (9) agree well in amplitudes of the head-wave, but differ essentially in phase of the wave-tail behind the cylinder. Thus, the nonlinear higher-order terms correct notably the qualitative behaviour of the solution observing immediately over submersed cylinder which moves horizontally near the free surface.

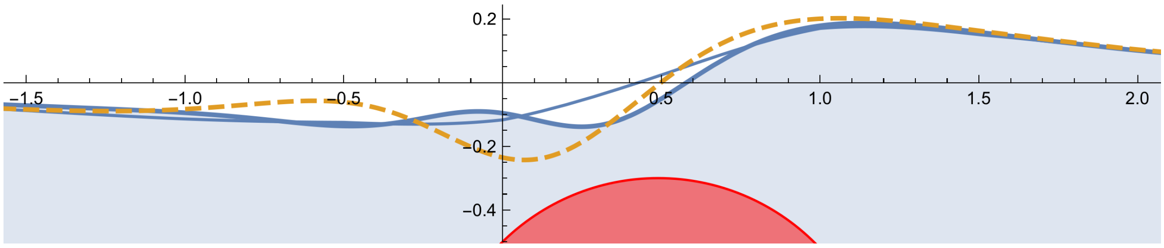

Fig 6: Horizontal motion of circular cylinder of radius $r = 0.7$ with $\lambda = 10$. Solid thick line – higher-order asymptotic solution (9), dashed line – leading-order solution (7), solid thin line – exact linear solution (10).

Conclusions The initial stage of non-stationary free surface flow generated by submerged circular cylinder starting from rest was investigated semi-analytically. Small-time asymptotic solution taking into account nonlinear terms was constructed and analysed. The impact of nonlinearity was clarified for two basic flow regimes: vertical submersion and horizontal motion of circular cylinder. It was demonstrated that higher-order nonlinear terms of asymptotic solution are capable to describe the formation of small-amplitude waves riding on the free surface.

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