Interaction of hydroelastic waves in ice cover with vertical walls

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The linear three-dimensional problem of uni-directional hydroelastic wave propagating in an infinite ice cover towards a vertical cylinder of an arbitrary cross section Γ in water of finite depth H, see figure 1, is solved by the vertical mode method. In two-dimensional problems of hydroelastic waves reflected from a vertical wall this method was used in [1]. Other methods to study both 2D and 3D problems of hydroelastic waves and their interactions with vertical structures were developed in [2-4]. It was noted in [5] than the eigenfunctions, which represent the vertical modes of the liquid flow between the flat sea bottom and a floating elastic ice sheet, are non-orthogonal in a standard sense and could be incomplete. The two-dimensional scattering problem for a crack was solved by a Green' function approach (see [5], section 3) and then by the eigenfunction expansion method (see [5], section 4), which is equivalent to the present method of vertical modes. It was reported in [9] that the vertical mode method is much simpler to use, and it also gives useful details of the ice deflection and the flow beneath the ice. However, to validate this mode solution, they solve the same problem by another method and demonstrated that these two solutions are identical. We followed the same idea and proved that the solutions of the problem for a vertical circular cylinder by the vertical mode method and by the method based on the Weber integral transform in the radial coordinate [3] are identical. In the present study, we generalize the method of vertical modes to any geometry of vertical walls. However, numerical results are still only for circular cylinders.

The vertical modes of ice sheet of constant thickness floating on water of finite depth were introduced in [6] and were successfully applied to two-dimensional problems of hydroelasticity without vertical boundaries in several papers.

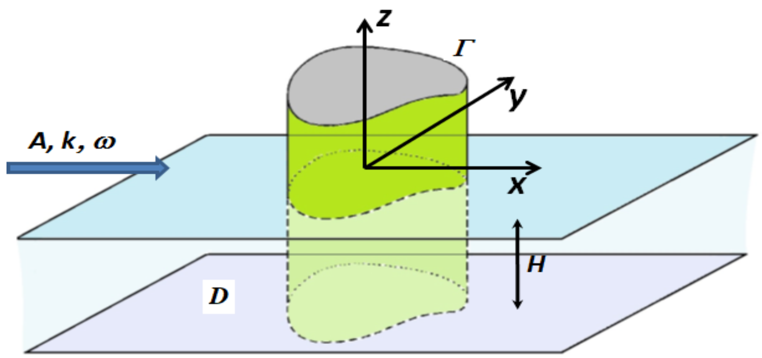

Fig. 1 Sketch of the problem and main notations.

Formulation of the problem

The flow and ice deflection are caused by an incident hydroelastic wave,

$$w_{inc}(\tilde{x}, \tilde{t}) = A\cos(k\tilde{x} - \omega\tilde{t}), \tag{1}$$

propagating in the positive x-direction, see figure 1, where A is the amplitude of the incident wave, k is the wavenumber and ω is the wave frequency. Here real and positive ω and k are related by the dispersion equation of hydroelastic waves. The linear problem of the incident hydroelastic wave interacting with a vertical bottom-mounted cylinder is formulated in non-dimensional variables (without $\tilde{.}$). The water depth H is taken as the length scale, $1/\omega$ as the time scale, A is the scale of the deflections and $AH\omega$ is the scale of the velocity potential of the flow. The ice deflection, $w(x,y,t)$, and the velocity potential, $\phi(x,y,z,t)$, are periodic in time,

$$w = \Re(W(x,y)e^{-it}), \quad \phi = \Re(\Phi(x,y,z)e^{-it}). \tag{2}$$

The complex potential $\Phi(x,y,z)$ satisfies the Laplace equation,

$$\nabla^2\Phi + \Phi_{zz} = 0, \qquad \nabla^2\Phi = \Phi_{xx} + \Phi_{yy}, \tag{3}$$

in the flow region, $-1 < z < 0$, $(x,y) \in D$. The plane $z = -1$ corresponds to the flat rigid bottom and the plane $z = 0$ corresponds to the ice-fluid interface. The potential Φ satisfies also the following boundary conditions,

$$\Phi_z = 0 \quad (z = -1, \quad D), \qquad \frac{\partial\Phi}{\partial n} = 0 \quad (-1 < z < 0, \quad \Gamma), \qquad \Phi_z = W \quad (z = 0, \quad D). \tag{4}$$

The equation of thin ice plate can be written in the form [14]

$$\frac{\partial^5\Phi}{\partial z^5} + \delta\frac{\partial\Phi}{\partial z} = q\Phi \quad (z = 0, \quad (x,y) \in D), \tag{5}$$

where $q = (\omega^2 H/g)(H/L_c)^4$, $\delta = (1 - \omega^2/\omega_0^2)(H/L_c)^4$, $L_c = (D_i/\rho g)^{1/4}$ is the characteristic length of the ice sheet, $\omega_0 = (\rho g/m)^{1/2}$ is the frequency of floating broken ice, m is the mass of the ice cover per unit area, $m = \rho_i h_i$, h_i is the ice thickness, ρ_i is the ice density, D_i is the rigidity coefficient of the ice sheet, $D_i = E_i h_i^3/[12(1-\nu^2)]$ for an elastic plate of constant thickness, E_i is the Young module of the ice, ν is the Poisson ratio, ρ is the water density and g is the gravitational acceleration. The condition at infinity follows from (1):

$$W \sim e^{i\text{æ}x} \qquad (x \to -\infty), \tag{6}$$

where $\text{æ} = kH$ is the non-dimensional wavenumber. The condition (6) is imposed where $x^2 + y^2 \to \infty$ if the vertical walls Γ do not extend to infinity. The three dimensionless parameters, δ, q and æ, are related by the dispersion relation

$$(\text{æ}^4 + \delta)\,\text{æ}\tanh(\text{æ}) - q = 0. \tag{7}$$

The conditions at the contact line, $z = 0$ and $(x,y) \in \Gamma$, between the ice cover and the surface of the cylinder can be complicated in practical problems. The present method is not sensitive to the type of these conditions. The method is demonstrated here for the ice cover being frozen to the vertical cylinder, which is modelled by the clamped conditions,

$$W = 0, \quad \frac{\partial W}{\partial n} = 0 \quad ((x,y) \in \Gamma). \tag{8}$$

Vertical mode method

By the method of separating variables, a product $\Phi(x,y,z) = W_n(x,y)f_n(z)$ satisfies equation (3) and the boundary conditions on the bottom (4_1) and the ice-water interface (5), if $f_n(z)$ is a nontrivial solution of the following spectral problem:

$$f_n'' - \text{æ}_n^2 f_n = 0 \qquad (-1 < z < 0), \qquad \frac{\mathrm{d}f_n}{\mathrm{d}z}(-1) = 0, \quad \frac{\mathrm{d}^5 f_n}{\mathrm{d}z^5} + \delta\frac{\mathrm{d}f_n}{\mathrm{d}z} = q f_n(0), \tag{9}$$

where æ_n is a root of the dispersion relation (7), $n = -2, -1, 0, 1, ...$, $\text{æ}_0 = \text{æ}$, and $W_n(x,y)$ is a solution of the equation

$$\nabla^2 W_n + \text{æ}_n^2 W_n = 0 \qquad ((x,y) \in D). \tag{10}$$

The functions $f_n(z)$ normalised by the condition $f'(0) = 1$ are $f_n(z) = \cosh[æ_n(z+1)]/(æ_n \sinh[æ_n])$. The vertical modes are orthogonal, $<f_j, f_n>= 0$, $<f_n, f_n>= Q_n$, where $j \neq n$ and the scalar product of two functions $F(z)$ and $G(z)$ defined in the interval $-1 \leq z \leq 0$, is

$$<F, G>= \int_{-1}^{0} F(z)G(z)\mathrm{d}z + \frac{1}{q}(F'''(0)G'(0) + F'(0)G'''(0)). \tag{11}$$

By algebra, $Q_n = (æ_n^2(æ_n^4+\delta)^2+q(5æ_n^4+\delta-q))/(2æ_n^2 q^2)$. For the imaginary roots of the dispersion relation, $æ_n = i\mu_n$, where $\mu_n > 0$ and $n \geq 1$, we have $\mu_n = \pi n - q(\pi n)^{-5} + O(n^{-6})$ as $n \to \infty$. Therefore, $Q_n = O(n^8)$ as $n \to \infty$. The conditions at infinity for equations (10) are

$$W_0 \sim e^{iæx}, \quad W_n \to 0 \qquad (x \to -\infty), \tag{13}$$

The solution of the original problem is given by the series

$$\Phi(x, y, z)) = \sum_{n=-2}^{\infty} W_n(x, y) f_n(z), \qquad W(x, y) = \sum_{n=-2}^{\infty} W_n(x, y). \tag{14}$$

The boundary condition for (10) are derived in local coordinates (s, n), where $n = 0$ on the vertical wall Γ and s is a curvilinear coordinate along the wall. By definition,

$$\lim_{n\to 0} <\frac{\partial \Phi}{\partial n}(x, y, z), f_k(z)>= \sum_{n=-2}^{\infty} \frac{\partial W_n}{\partial n} <f_n, f_k>= \frac{\partial W_k}{\partial n}(s)Q_k,$$

on the other hand the limit is equal to

$$\lim_{n\to 0} \left[\int_{-1}^{0} \frac{\partial \Phi_m}{\partial n}(x, y, z) f_k(z)\mathrm{d}z + \frac{1}{q}\left(\frac{\partial^3}{\partial z^3}\left(\frac{\partial \Phi}{\partial n}\right)(x, y, 0) f_k'(0) + \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial n}\right)(x, y, z) f_k'''(0) \right) \right],$$

where the limit of the integral is zero. The boundary condition is

$$\frac{\partial W_k}{\partial n}(s) = \frac{Q(s)}{qQ_k} + \frac{æ_k^2}{qQ_k}\frac{\partial W}{\partial n}(s), \tag{15}$$

where the functions $Q(s)$ and $(\partial W/\partial n)(s)$ are defined along the contact line between the ice and the vertical walls, $Q(s)$ is the shear force with the scale $D_i A H^{-3}$ and $\partial W/\partial n(s)$ is the slope of the ice plate at the vertical wall. The functions $Q(s)$ and $(\partial W/\partial n)(s)$ are to be determined by using the conditions at the contact line.

For clamped conditions, $(\partial W/\partial n)(s) = 0$ and

$$Q(s) = \sum_{m=0}^{\infty} \mu_m g_m(s), \qquad W_k(s) = (qQ_k)^{-1} \sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \mu_m C_k^{mp} g_p(s) \qquad \int_\Gamma g_m(s) g_j(s)\mathrm{d}s = \delta_{mj},$$

where $g_m(s)$ make a complete set of functions along the contact line, $W_0(s) = W_{0d}(s) + W_{0r}(s)$, where W_{0r} is the radiation part of the deflection. The clamped condition, $W(s) = 0$ gives the following algebraic system with a symmetric matrix for unknown coefficients μ_m, where $p \geq 0$,

$$\sum_{m=0}^{\infty} \mu_m \left(\sum_{n=-2}^{\infty} (qQ_n)^{-1} C_n^{mp} \right) = -\int_\Gamma W_{0d}(s) g_p(s)\mathrm{d}s. \tag{16}$$

Other conditions at the contact line are treated in a similar way.

For a circular cylinder, s is the angular coordinate, $0 \leq s < 2\pi$, $g_m(s) = \nu_m \cos(s)$, $C_n^{mp} = 0$ for $m \neq p$ and

$$C_n^{mm} = \frac{H_m^{(1)}(æ_n B)}{æ_n H_m^{(1)\prime}(æ_n B)},$$

where $B = b/H$ is the non-dimensional radius of the cylinder.

The strain distribution around the cylinder in an incident wave is important to estimate possibility for ice to be broken due to the wave-structure interaction. The yield strain for ice is estimated as 8×10^{-5}, see [4]. On the contact line of the cylinder frozen in ice, only the radial strain component, $\epsilon_r(s, t) = 0.5 h_i w_{rr}(b, s, t)$, is not equal to zero. The amplitude of the radial strain as a function of the polar angle θ is shown in Figure 2 for a circular cylinder of radius 5 m, wave amplitude of 1 cm and different wave length It is seen that the incident wave of amplitude 1 cm and length 67 meters is strong enough to break the ice connection to the cylinder.

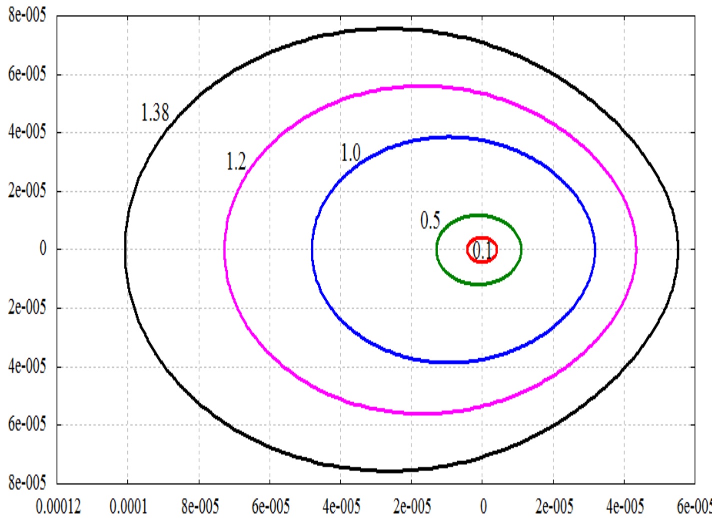

Fig. 2 Radial strains at the contact line in polar coordinates for $kH = 0.1, 0.5, 1.0, 1.2, 1.38$

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