Backward Waves through Array of Rectangular Columns

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1 INTRODUCTION

Backward waves are waves whose phases propagate in the opposite direction to energy propagation. Such waves occur because signs of phase and group velocities become reversed and are usually observed in a negative refractive index medium. The study of backward waves and the negative refractive index has attracted much attention since Smith et al. (2000) and Pendry (2000) showed a feasible technique of a negative index material in electromagnetic waves. Soon afterwards, this concept has been extended from optics to other kind of waves, such as acoustic or plasma waves. In the context of surface water waves, Hu et al. (2004) theoretically and experimentally demonstrated a super lensing effect by using a number of circular cylinders which were designed to attain the negative refraction in gravity-capillary waves. They reported that surface waves are modulated by multiple Bragg scatterings when propagating in a periodic structure.

In this paper, we present another method for realizing structures to attain negative refractive index for gravity dominant water waves in shallow and deep water. We theoretically obtain a condition where backward waves occur by an array of rectangular columns. In order to derive the condition, an analogy between water waves in a channel and electric waves in an electric circuit is utilized. Now attention is not focused on the columns themselves, but the space among columns where waves propagate. Then, the space among columns is regarded as the water channel. Once an equivalent circuit model of the water channel is obtained, we can easily analyze complicated models based on the channel, and get characteristics (e.g. impedance and refractive index) of waves through the channel. Here, the space among four rectangular columns is especially considered, and thus it is treated as a cross joint water channel. The refractive index in the cross joint water channel is obtained by equivalent circuit analysis for both shallow and deep water cases. We identify the condition where the refractive index becomes negative value, which is called Negative Index Metamaterial (NIM) condition. We show that the phase velocity becomes also negative value under the NIM condition, in contrast to positive group velocity. Namely, backward waves are observed among the array of rectangular columns.

Numerical experiments are carried out by a finite element method to validate the proposed condition. The linear potential theory is assumed for both theoretical formulation and numerical experiment. To obtain the NIM condition, we only consider the progressive wave mode, but higher wave modes are taken into account in computations. Parameters of the water channel (i.e. rectangular columns) are designed to match the impedance to an open space with the NIM condition.

2 THEORY

Let us describe an equivalent circuit model to waves in a rectangular water channel (Mochizuki et al., 1990, Iida et al., 2017). First, we consider a one directional water channel which is equivalent to a space between two rectangular columns. Here ℓ is the size of the channel, b is the width and d is the depth of the columns. Subscripts s and d denote the values in shallow and deep water cases, respectively, so that the column depth d_s is equal to the water depth h_s, and d_d is equal to half wave length $\lambda/2$. We consider a linearized potential flow problem, which assumes incompressible and inviscid flow with irrotational motion. Then, the x-components of the continuity equation and Euler equation are given as

$$-\frac{\partial P}{\partial x} = \rho\frac{\partial u}{\partial t},\ -\frac{\partial u}{\partial x} = \frac{k_x^2}{\rho\omega^2}\frac{\partial P}{\partial t} \tag{1}$$

where P is the pressure, ρ is the fluid density, ω is the circular frequency, u is the x-component of velocity and k_x are the wave number for x-direction. The wave number is given by following relations

$$k_{xs} = \sqrt{\frac{\omega^2}{gd_s} - \left(\frac{n\pi}{b}\right)^2}, \quad k_{xd} = \sqrt{\frac{\omega^4}{g^2} - \left(\frac{n\pi}{b}\right)^2} \tag{2}$$

where k_{xs} and k_{xd} are the wave numbers in shallow and deep water and n is integer which denotes the mode of waves. $\omega_{cs} = \sqrt{gd_s}(n\pi/b)$ and $\omega_{cd} = \sqrt{g(n\pi/b)}$ are cut-off frequencies. Here we consider only progressive wave mode $n = 0$ and higher modes are discarded to simplify the problem. Introducing flow rate $Q = bdu$, eq.(1) can be deformed as

$$\frac{\partial P}{\partial x} = -i\omega L_1 Q, \ \frac{\partial Q}{\partial x} = -i\omega C_1 P \tag{3}$$

where solutions are assumed sinusoidal in time and L_1 and C_1 are equivalent inductance and capacitance in one directional water channel, which hold $L_{1s,d} = \rho/bd$, $C_{1s} = b/\rho g$ and $C_{1d} = bd\omega^2/\rho g^2$. Eq.(3) is equivalent to the telegraphic equations of loss less circuit. From eq.(3), pressure P and flow rate Q are analogized to voltage V and electric current I. Then, the circuit analysis is applied and the characteristics of waves in the channel are determined as

$$Z_{1s} = \sqrt{\frac{L_{1s}}{C_{1s}}} = \frac{\rho}{b}\sqrt{\frac{g}{d_s}}, \quad k_{1s} = \omega\sqrt{L_{1s}}\sqrt{C_{1s}} = \frac{\omega}{\sqrt{gd_s}} \tag{4}$$

$$Z_{1d} = \sqrt{\frac{L_{1d}}{C_{1d}}} = \frac{\rho g}{bd_d\omega}, \quad k_{1d} = \omega\sqrt{L_{1d}}\sqrt{C_{1d}} = \frac{\omega^2}{g} \tag{5}$$

where Z_1 is the characteristic impedance. Eqs. (4) and (5) indicate that the wave numbers obtained by the circuit analysis are equivalent to the ones in eq.(2) when $n = 0$.

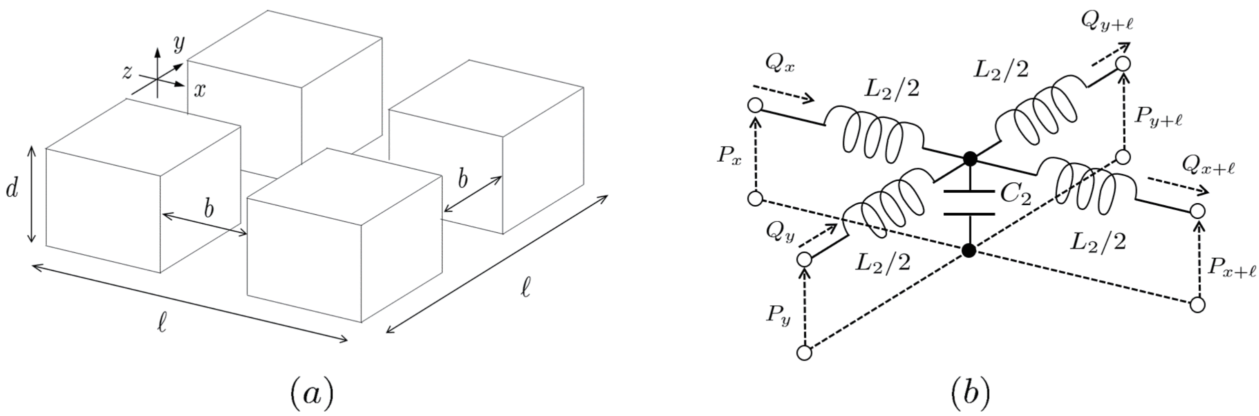

Fig. 1 (a) View of cross joint channel (four rectangular columns) (b)Equivalent electric circuit model of the cross joint water channel (a).

Once equivalent circuit parameters are obtained, this model is extended to the cross joint water channel. Fig.1(a) shows a view of the cross joint water channel which is composed of four rectangular columns. This cross joint water channel is represented by an equivalent electric circuit model and distributed constant circuit model. Here we only show the equivalent circuit model in Fig.1(b), and the distributed constant circuit model is shown in Iida et al. (2017). At a junction of the channel, the flow rate which can propagate to becomes double compared to the one directional channel, and this effect is reflected to the equivalent capacity. Both circuit models are solved by Kirchhoff's circuit laws. Since both circuit models are equivalent, properties of the cross joint water channel are determined as

$$L_2 = \frac{2Z_1}{\omega\ell}\tan\frac{k_1\ell}{2}, \quad C_2 = \frac{2}{\omega\ell Z_1}\sin k_1\ell \tag{6}$$

where L_2 and C_2 are equivalent inductance and capacitance in the cross joint water channel. Therefore, the characteristics of waves in the cross joint water channel are obtained as

$$Z_2 = \sqrt{\frac{L_2}{C_2}} = Z_1 \sqrt{\frac{\tan k_1\ell/2}{\sin k_1\ell}} \tag{7}$$

$$n_2 = \frac{k_2}{\omega} = \sqrt{L_2}\sqrt{C_2} = \frac{2}{\omega\ell}\sqrt{\tan\frac{k_1\ell}{2}}\sqrt{\sin k_1\ell} \tag{8}$$

where n_2 is the refractive index in the cross joint water channel. Since the impedance Z_1 and wave number k_1 are given as functions of parameters of the one directional channel, the characteristics of the cross joint water channel are determined. Eq.(6) indicates that if the length ratio $k_1\ell$ satisfies $\pi + 2m\pi < k_1\ell < 2\pi + 2m\pi \quad (m = 0, 1, 2, \cdots)$, both equivalent inductance and capacitance become negative. As a result, the refractive index attains negative value as shown in eq.(8). This condition is a negative index metamaterial condition (NIM condition) for realizing the negative index by the cross joint water channel. As eq.(8), waves must have dispersion to attain the negative refractive index. Under the NIM condition, the phase velocity becomes

$$c_p = \frac{\omega}{k_2} = \frac{1}{\sqrt{L_2}\sqrt{C_2}} = -\frac{\omega\ell}{2\sqrt{\tan(k_1\ell/2)\sin k_1\ell}} < 0 \quad \text{on NIM} \tag{9}$$

Here the relation $\sqrt{a}\sqrt{b} = -\sqrt{ab}$ where $a < 0,\ b < 0$ is used. It indicates that the phase velocity becomes also negative under the NIM condition. On the other hand, the group velocity becomes positive as shown by the following equation

$$c_g = \frac{\partial\omega}{\partial k_2} = -\frac{1}{\sqrt{2}\cos k_1\ell/2}\left(\frac{\partial\omega}{\partial k_1}\right) > 0 \quad \text{on NIM} \tag{10}$$

Positive group velocity means that the energy propagates to the positive direction, and the negative phase velocity indicates the phase moving in the opposite direction to energy propagation, that is, backward waves occur.

3 DESIGN AND RESULT

Table 1 Optimized parameters of water channel for shallow and deep water cases.

	d	ℓ	b	ω	Z_2'	n_2'
Shallow water	0.5	11.07	8.82	1.07	1.00	-0.24
Deep water	$\lambda/2$	8.58	6.86	2.46	1.00	-0.24

We design the water channel for realizing the negative refractive index and backward waves. Design variables are channel length ℓ, width b and frequency ω. These variables are optimized to match the impedance between the channel and an open space $Z_2' = Z_2/Z_0 = 1.0$ with the NIM condition where refractive index $n_2' = n_2/n_0$ becomes negative. Optimized parameters for shallow and deep water waves are shown in Table 1. Here the water depth is sufficiently smaller than the wave length $20h_s = 20d_s < \lambda$ in shallow water, and deeper than the wave length $h_d > \lambda/2 = d_d$ in deep water, respectively. Since the impedance matching is satisfied, the cross joint water channel is treated as a unit structure and a number of channels can be periodically cascaded. Then, this structure is regarded as the infinite array of rectangular columns.

In order to measure wave patterns among columns, numerical experiments are carried out. We use a finite element method based software COMSOL Multiphysics to solve 3-dimensional boundary value problem. Fluid domain is governed by Laplace's equation and the linearized free surface condition is applied on the free surface. No flux condition is applied at the interface of columns and bottom boundaries. Waves are incident from left to right boundaries. Here, we show the resultant wave pattern at oblique angle $\beta = 0°$ and $3°$ in deep water as in Fig.2(a) and (b). Other cases, wave patterns in shallow water and different incident angles will be shown in the Workshop. Arrows denote the direction of the phase moving. As arrows indicate, waves in the array of columns are propagating to the right to left, which is opposite to the direction of incident waves. In addition, waves are refracted at the negative angles at the interface of the array in Fig.2(b). The refractive angle in computation is $\gamma \simeq -24°$ and it is underestimated compared to the angle obtained by Snell's law $\gamma = -12.4°$. It is because we only consider progressive waves which propagate along the x-axis. If incident waves have angles, effects of higher wave modes become more and more important. In fact, if the oblique angle increases, this array does not work as the negative index structure. Nevertheless, backward waves are observed in the numerical experiment as mentioned by the proposed theory. This study is yet a stage of fundamental research, however, we expect it might be meaningful to understand wave phenomenon around the structures and develop innovative technologies because periodical array of columns are commonly used in the ocean development industry.

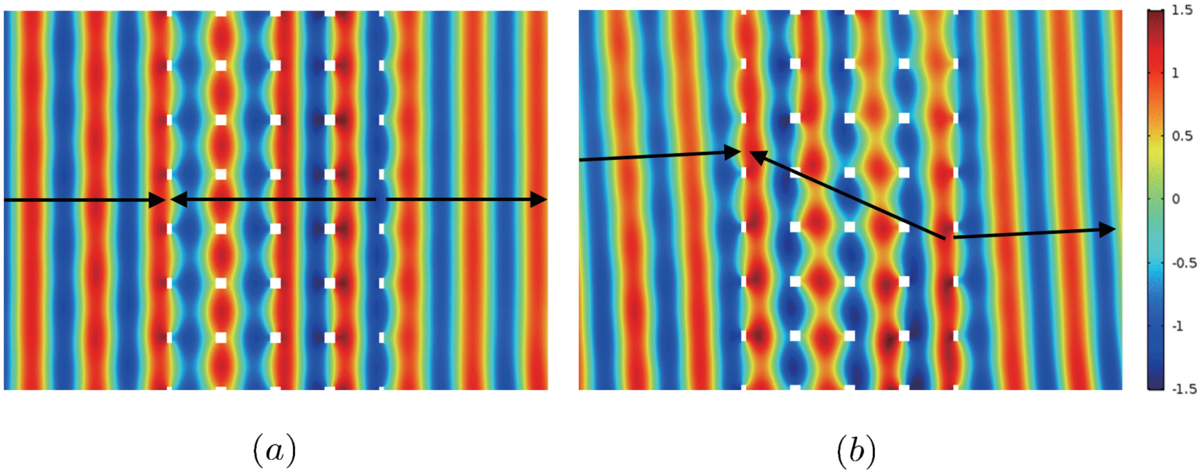

Fig. 2 Wave pattern in deep water at frequency $\omega_d = 2.47$ ($\lambda = 10.12$) with (a) incident angle $\beta = 0°$ and (b) incident angle $\beta = 3°$. Arrows denote wave lays which phases move along by.

4 CONCLUSION

The condition for realizing the negative refractive index and backward waves by an array of the rectangular columns are proposed in both shallow and deep water. The analogy between water waves in the channel and electric waves in the circuit is utilized with progressive waves to obtain the characteristics of waves among four rectangular columns (thus we regard the space as the cross joint water channel). The channels are designed to match the impedance with negative index condition, and thus they are periodically cascaded as the array of the columns. Numerical experiments are carried out to validate the proposed condition and we show the deep water case at incident angle $\beta = 0°$ and $3°$. Resultant wave pattern shows that the refractive index and backward waves are observed when the incident angle is not large. Other cases will be shown in the Workshop.

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