Wave Diffraction by Multiple Vertical Cylinders: The Nonlinear Shallow Water Wave Equations

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Abstract

This study is concerned with the diffraction of nonlinear waves by in-line, fixed, vertical, circular cylinders in shallow water. The solitary and conical wave-structure interaction problem is studied by use of the nonlinear Level I Green-Naghdi equations and the nonlinear generalised Boussinesq equations. The problem is also solved by the linearized version of the equations to study the nonlinear effects. Results are compared with laboratory measurements.

Introduction

Interaction of nonlinear waves with vertical cylinders is an important applied problem that has received significant attention. Some of the notable approaches include Morison's equation, the linear diffraction solution of MacCamy & Fuchs (1954) and the nonlinear solution of Faltinsen et al. (1995) by solving nonlinear potentials. Vertical cylinders have vast applications in shallow water. Yet, there are very limited studies on the interaction of long waves with multiple cylinders in shallow water.

IN shallow water, dispersion and nonlinearity can significantly modify the wave dynamics, and hence influence the wave-structure interaction problem. Our goal in this work is to develop theoretical models based on nonlinear shallow water wave theories to study the problem of wave interaction with multiple vertical cylinders. We use two theories that are well-known in capturing the nonlinearity and dispersion in shallow water, namely the Green-Naghdi (GN) equations and the Boussinesq equations.

We also develop a model to study the wave-structure interaction problem by use of the linearised version of these equations. The linearised equations are obtained by removing the nonlinear terms of the GN and Boussinesq equations. Results of the nonlinear models are compared with each other, with the results of the linear equations, and with laboratory experiments when available.

The Water Wave Theories

We assume a homogeneous, inviscid and incompressible fluid. A right-handed Cartesian coordinate system (x_1, x_2, x_3) is used, whose origin is at the still-water level (SWL). x_1 is to the right and x_2 is pointing upward, opposite to the gravitational acceleration direction. Soliatry and cnoidal waves are generated at the left-hand side and propagate in the positive x_1 direction. The circular cylinders are fixed and extend from the free surface to the flat seafloor. Cylinders diameter, D, is constant; they are in a row in x_1 direction, and in general there are N of them, where $N = 1, 2, 3, \quad$. The equations are given in dimensionless form using ρ, g and h, fluid density, gravitational acceleration and water depth, respectively, as a dimensionally independent set.

Boussinesq Equations

Boussinesq equations are developed for an irrotational flow and by defining the velocity potential. In general, the Boussinesq equations are derived assuming that parameters $\sigma = H/h$ and $\epsilon = h/\lambda$, namely nonlinearity and dispersion, are small, where H, and λ are the wave height and wavelength, respectively. Various versions of Boussinesq-class equations are given in the literature. Here, we use the equations given by Wu (1981), namely the generalised Boussinesq (gB) equations:

$$\eta_{,t} + \boldsymbol{\nabla} \cdot \{(1+\eta)\, \boldsymbol{\nabla}\phi\} = 0\,, \tag{1}$$

$$\phi_{,t} + \frac{1}{2}||\boldsymbol{\nabla}\phi||^2 + \eta = \frac{1}{3}\boldsymbol{\nabla}\phi_{,t}\,, \tag{2}$$

$$P(x_1, x_2, x_3, t) = \eta - x_2 + \left(x_2 + \frac{1}{2}x_2^2\right) \boldsymbol{\nabla} \cdot \boldsymbol{V}_{,t}\,, \tag{3}$$

where ϕ is the layer-mean velocity potential, η is the surface elevation measured from the still-water level, and $\boldsymbol{\nabla}$ is the gradient operator. P is the total pressure and $\boldsymbol{V} = (u_1, u_2, u_3)$ is the three-dimensional velocity vector. The subscripts after comma indicate partial differentiations. Here, it is assumed that $O(\sigma) = O(\mu^2) < 1$, where $\mu = kh = 2\pi\epsilon$. Therefore, the gB equations are mostly applicable when $Ur = \sigma/\mu^2$ is of $O(1)$.

The Green-Naghdi Equations

The nonlinear GN equations, an alternative to the perturbation-based methods, can be derived by making a single assumption about the kinematics of the incompressible fluid flow (Green & Naghdi (1976)). That is, by prescribing the distribution of the vertical velocity along the water column. In the Level I GN equations, for example, a linear distribution of the vertical velocity over the water column in assumed. In high-level GN equations, the variation of the vertical velocity is defined by high-order polynomials. In the absence of perturbation expansion in the derivation of the GN equations, the nonlinear boundary conditions and the integrated conservation laws are satisfied exactly. No *a priori* assumption about the wave field is required. Hence, limitations of the GN equations is implicit and must be determined through comparison with laboratory experiments.

In dimensionless form, the complete Level I GN equations are given by

$$\eta_{,t} + \boldsymbol{\nabla} \cdot \{(1+\eta-\alpha)\boldsymbol{V}\} = \alpha_{,t}\,, \tag{4}$$

$$\dot{u}_1 + \eta_{,x_1} + \hat{p}_{,x_1} = -\tfrac{1}{6}\{[2\eta+\alpha]_{,x_1}\ddot{\alpha} + [4\eta-\alpha]_{,x_1}\ddot{\eta} + (1+\eta-\alpha)[\ddot{\alpha}+2\ddot{\eta}]_{,x_1}\}\,, \tag{5}$$

$$\dot{u}_3 + \eta_{,x_3} + \hat{p}_{,x_3} = -\tfrac{1}{6}\{[2\eta+\alpha]_{,x_3}\ddot{\alpha} + [4\eta-\alpha]_{,x_3}\ddot{\eta} + (1+\eta-\alpha)[\ddot{\alpha}+2\ddot{\eta}]_{,x_3}\}\,, \tag{6}$$

$$u_2(x_1, x_3, t) = \dot{\alpha} + \tfrac{(x_2+1-\alpha)}{(\eta+1-\alpha)}\left(\dot{\eta} - \dot{\alpha}\right)\,, \tag{7}$$

$$P_I(x_1, x_3, t) = \tfrac{1}{6}\left(1+\eta-\alpha\right)^2\left(\ddot{\alpha} + 2\ddot{\eta} + 3\right) + \hat{p}\left(1+\eta-\alpha\right)\,, \tag{8}$$

$$\bar{p}(x_1, x_3, t) = \tfrac{1}{2}\left(1+\eta-\alpha\right)\left(\ddot{\alpha} + \ddot{\eta} + 2\right) + \hat{p}\,, \tag{9}$$

where $\alpha(x_1, x_3, t)$ is the elevation of the bottom of the fluid sheet, P_I is the integrated (over the water column) pressure, $\bar{p}$ is the pressure on the bottom surface (α), $\hat{p}(x_1, x_3, t)$ is the pressure on the top surface of the fluid sheet (assumed atmospheric here), and superposed dot is the two-dimensional material time derivative.

The Linearized Equations

The linear version of the gB and GN equations can be obtained by expanding all terms and removing the nonlinear terms. The linear version of the gB and GN equations are identical for a stationary seafloor ($\alpha, t = 0$, see Ertekin (1984) for details) and are given by

$$\eta_{,t} + \boldsymbol{\nabla} \cdot \boldsymbol{V} = 0\,, \tag{10}$$

$$u_{1,t} + \eta_{,x_1} = \frac{1}{3}\left[(u_1)_{,x_1x_1} + (u_1)_{,x_3x_3}\right]_{,t} = \frac{1}{3}\Delta u_{1,t}\,, \tag{11}$$

$$u_{3,t} + \eta_{,x_3} = \frac{1}{3}\left[(u_3)_{,x_1x_1} + (u_3)_{,x_3x_3}\right]_{,t} = \frac{1}{3}\Delta u_{3,t}\,. \tag{12}$$

The Numerical Wave Tank

At $x_1 = 0$, a wavemaker is located to generate solitary or conical waves. On the right-hand side, Orlanski's condition is used for the wave absorbing boundary. The governing equations are solved by second-order central-difference schemes in space, and by the modified Euler method for time marching.

To facilitate the use of the finite difference scheme in solving the equations for the curved boundaries, an elliptical grid generation technique is used to map the physical domain into the computational domain with a boundary fitted curvilinear coordinate.

The wave-induced horizontal force on the vertical cylinders is obtained by integrating the averaged (over the cylinder height) pressure around the circumference of the cylinders. For the GN equations, the averaged pressure is given by Eq. (8). For the gB equations, however, this is obtained by integrating P, Eq. (3), over the water column. The gB integrated pressure is then given by

$$P_I = \frac{1}{2}(1+\eta)^2 + \frac{1}{6}(1+\eta)\left(\eta^2 + 2(\eta-1)\right)\boldsymbol{\nabla}\cdot\boldsymbol{V}. \tag{13}$$

To determine the wave-induced moment, instantaneous pressure distribution around vertical cylinders is required. For the gB equations, the pressure distribution is given by Eq. (3). In deriving the GN equations, however, pressure is integrated over the water column, Eq. (8). In the absence of an explicit relation for the pressure distribution in GN theory, we assume a linear (total) pressure distribution between $\hat{p}$ on the top surface, and $\bar{p}$ (Eq. (9)) on the bottom surface, i.e.,

$$P(x_1, x_2, x_3, t) = \frac{1}{2}(\eta - \alpha - x_2)(\ddot{\alpha} + \ddot{\eta} + 2) + \hat{p}\,. \tag{14}$$

Comparisons show close agreement between results of Eqs. (3) and (14), the gB and GN pressure distribution equations, respectively. The error associated to assuming Eq. (14) for the GN pressure distribution in the vertical direction can be determined, and it is found to be less than 1.8% for solitary wave and 0.29% for cnoidal waves for the cases studied here.

Results and Discussion

Sample snapshots of interaction of cnoidal waves with three in-line vertical cylinders, obtained by the GN equations, is shown in Fig. 1. The spacing, S, is defined as the outer distance between cylinders. The wave-induced force on the cylinders is presented in dimensionless form as $\bar{F} = \frac{F}{\rho g h^2 R}$, where R is the cylinder radius. Time is given dimensionless as $\bar{t} = t\sqrt{g/h}$.

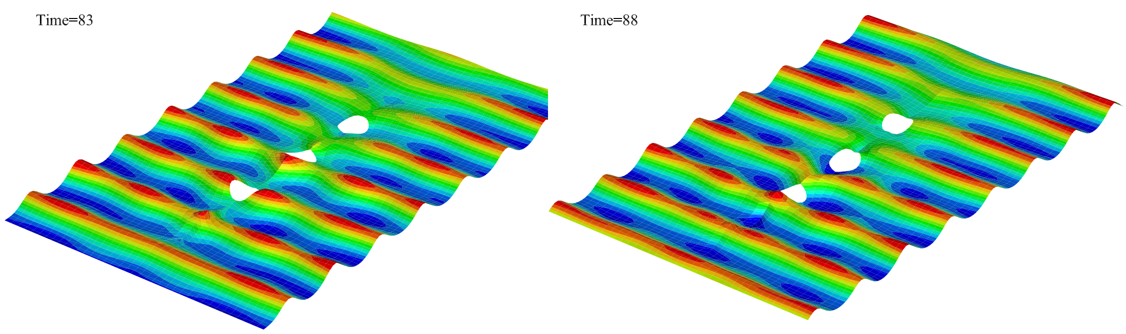

Figure 1: 3-D snapshots of cnoidal wave surface elevation around three cylinders, obtained by the GN equations. $H = 0.2h, \lambda = 9.0h, D = 4.0h$ and $S = 2.0D$.

The solitary wave-induced force on three in-line cylinders, calculated by the GN and the gB equations, are shown in Fig. 2. The gB equations predict slightly larger force on the cylinders. This is consistent with comparisons with laboratory experiment results, where the GN equations are in closer agreement. Due to the shielding effect of the first cylinder, the maximum forces on the second and

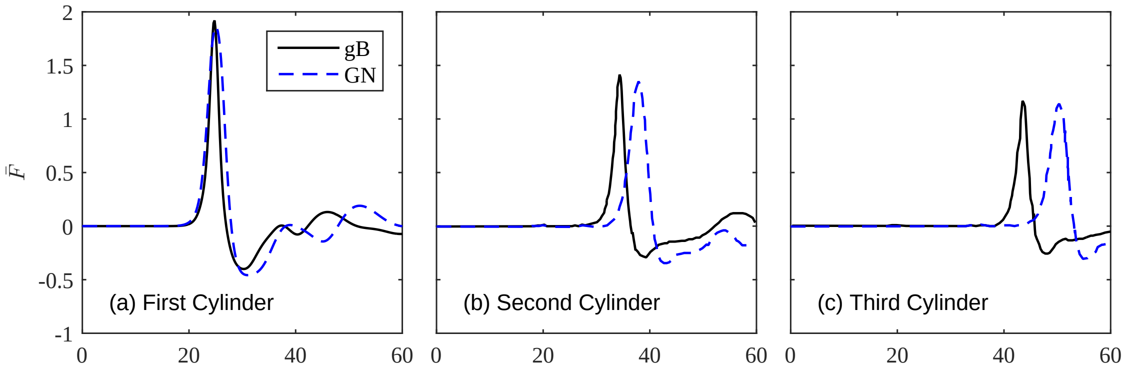

Figure 2: Solitary wave force on three in-line vertical cylinders, calculated by the Boussinesq (gB) and the GN equations; $A = 0.5h, D = 4.0h$, and $S = 1D$, where A is the solitary wave amplitude.

third cylinders are smaller than that on the first cylinder. The wave force on the first cylinder in the three-cylinder case is slightly larger than the wave force on a single cylinder. This will be discussed further at the workshop.

Cnoidal wave forces on two in-line cylinders, calculated by the nonlinear shallow water wave equations, and by the linearized equations, are shown in Fig. 3. The linear solution has underestimated the force on the cylinders. The wave-induced force on the downwave cylinders is smaller than that on the upwave cylinder. In some cases, however, the loads on the second cylinder are found to be larger than those on the first cylinder. The effect of the neighboring cylinders is more complex for cnoidal waves compared with the solitary wave, and varies nonlinearly with the wavelength and the cylinder spacing.

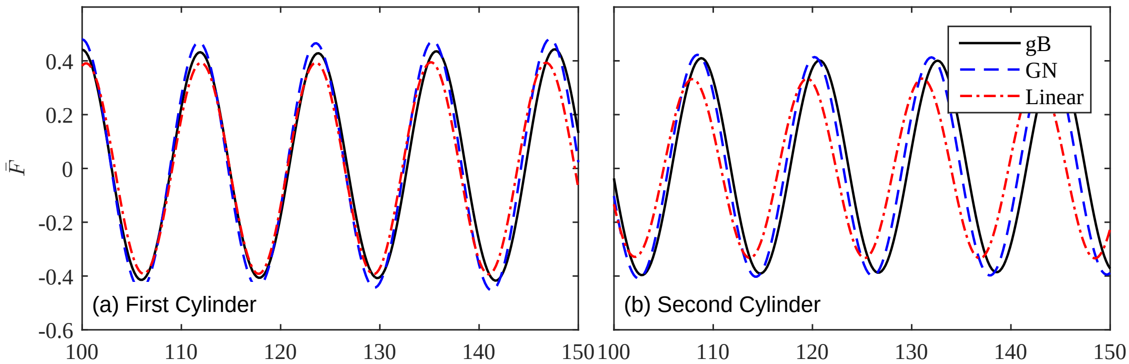

Figure 3: Periodic wave force on two in-line vertical cylinders, calculated by the Boussinesq (gB), the GN, and the linearized equations; $H = 0.2h, \lambda = 11h, D = 4.0h$, and $S = 2D$.

Comparisons show that the wave nonlinearity can significantly increase the forces. Further results and discussion on wave runup, wave-induced force and moment, and comparisons with laboratory experiments will be presented at the workshop.

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