Extraordinary transmission past cylinders in channels

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1 Introduction

The term extraordinary transmission originates from the optical literature – see Ebbesen *et al.* (1998). It describes the enhanced transmission of light through thin metallic screens perforated with periodic arrays of sub-wavelength holes, to orders of magnitude in excess of that expected from the consideration of transmission through a single hole. The term extraordinary transmission has been extended into other physical disciplines to encompass a broad range of settings which gives rise to enhanced transmission of wave energy through small apertures.

In water waves Evans & Porter (2017) showed that total transmission of waves is possible through multiple thin screens spanning a channel each of which contained a small aperture. It was shown that only two screens, sufficiently widely spaced, were required for total transmission which persists for vanishingly-small apertures. This result was provided in closed-form based on a small-gap approximation to certain integral equations that arise from the exact formulation of a solution to the problem. Those approximations were confirmed by full numerical solutions.

That work led us to speculate that total transmission should also be possible for cylinders in channels which extend a sufficient length of the channel to produce the multiple-scattering effects that underpin the results of Evans & Porter (2017). We have considered two such geometries: cylinders of circular and rectangular cross-sections. In both cases, interest centres on the transmission of wave energy when the cylinders are close to touching the wall.

2 Circular Cylinders

The problem of transmission of incident waves by a circular cylinder on the centreline of a channel is considered by Linton & Evans (1992) and its long history is described in that paper. Of particular interest is Fig. 8 of Linton & Evans (1992) which shows frequencies at which there is zero reflection for a cylinder of radius a occupying 90% of the width, d, of the channel.

We have repeated the numerical calculations of Linton & Evans (1992) and Fig. 1(a) shows how the modulus of the reflection coefficient varies over a range of values of cylinder occupancy up to 99%. It can be seen that total transmission ceases to exist as the cylinder approaches the channel walls, beyond a ratio of $a/d \approx 0.92$.

Separately, experiments were performed to measure reflection and transmission from a 53mm gap between two vertical half-cylinders mounted against the sides of a 430mm wide wave tank with a water depth of 700mm. The arrangement is shown in Fig. 2, in which the half-cylinders are mirrored in the tank's glass walls. Resistive wave gauges on the centreline of the tank towards the wavemaker on one side, and towards the wave absorber on the other, recorded water surface elevations first in the absence of the half-cylinders, and then again with them in place. Measurements were made over a range of frequencies with a constant wave steepness $kA = 0.05$. In each case data collection was complete before reflections from the wavemaker or from the wave absorber arrived at the wave gauges.

In Fig. 1(b) we have plotted curves of modulus of reflection and transmission coefficients against wavenumber predicted by the theory and the experimental results for $a/d = 0.8844$. Overall, the agreement is excellent and confirms that a high proportion of wave energy is able to propagate through a very small gap with loss due to nonlinear surface and viscous wall effects.

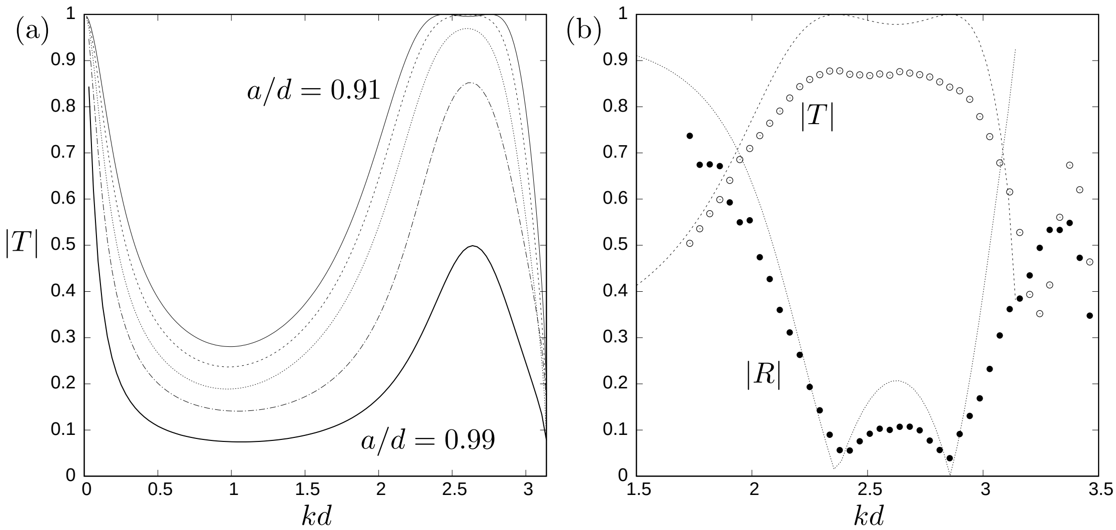

Figure 1: In (a), $|T|$ as a/d increases through 0.91, 0.93, 0.95, 0.97, 0.99. In (b) experimental results (circles) and theory (lines) for $a/d = 0.8844$.

The solution of Linton & Evans (1992) is constructed by expanding the wave field in terms of multipole potentials which are modified for the channel walls. This approach suffers from increasingly poor numerical convergence as the cylinder approaches the channel walls suggesting that it is not an appropriate tool to analyse the results in the limit at $a/d \to 1$. An alternative approach is to use the method of matched asymptotic expansions. Schnitzer (2016) describes the method applied to a problem with an identical geometry but in a different physical setting, which can be adapted to the current problem.

Since the numerical evidence suggests the absence of extraordinary transmission when $a/d \to 1$, we have not yet investigated the full extent of the implementation of matched asymptotics for this problem but hope to report on this at the workshop.

3 Rectangular Cylinders

Dupont *et al.* (2012) have observed experimentally that extraordinary transmission can occur when a rectangular cylinder spans almost all of the width of a narrow channel. More recently Meylan *et al.* (2017) have considered the problem numerically. Our aim here is to provide a closed-form expression for the transmitted wave energy under the assumptions of linearised theory and use this to prove that total transmission of wave energy is possible for vanishingly-small gaps between the walls of the cylinder and the channel. Our primary approach is be based on matched asymptotic expansions and extensive use is made of Newman *et al.* (1984) in which a similar problem is considered.

A fluid occupies a channel $|y| < d$, $-\infty < x < \infty$, $-h < z < 0$ in which a solid vertical cylinder of constant rectangular cross-section $|y| < a$, $|x| < b$ extends through the fluid, $-h < z < 0$. We assume that $\epsilon = 1 - a/d \ll 1$ such that the gap between the walls of the cylinder and the channel are small. The flow is described by a velocity potential which can be written as $\Re\{\phi(x, y) \cosh k(z + h)\mathrm{e}^{-\mathrm{i}\omega t}\}$ where k is related to the depth and frequency by $\omega^2/g = k \tanh kh$. It follows that ϕ satisfies

$$(\nabla^2 + k^2)\phi = 0 \tag{1}$$

in the fluid with no-flow conditions ($\partial_n\phi = 0$) on all solid walls of the cylinder and the channel. We assume that a plane wave is incident from $x = +\infty$ with $kd < \pi$ (below the first channel cut-off) so that in the far-field

$$\phi(x, y) \sim \begin{cases} \mathrm{e}^{-\mathrm{i}kx} + R\mathrm{e}^{\mathrm{i}kx}, & x \to \infty \\ T\mathrm{e}^{-\mathrm{i}kx}, & x \to -\infty \end{cases}$$

and R and T represent the reflection and transmission coefficients. The symmetry of the geometry allows us to write $\phi = \frac{1}{2}(\phi^s + \phi^a)$ where $\phi^s(x, y) = \phi^s(-x, y)$ and $\phi^a(x, y) = -\phi^a(x, y)$ and now we are only

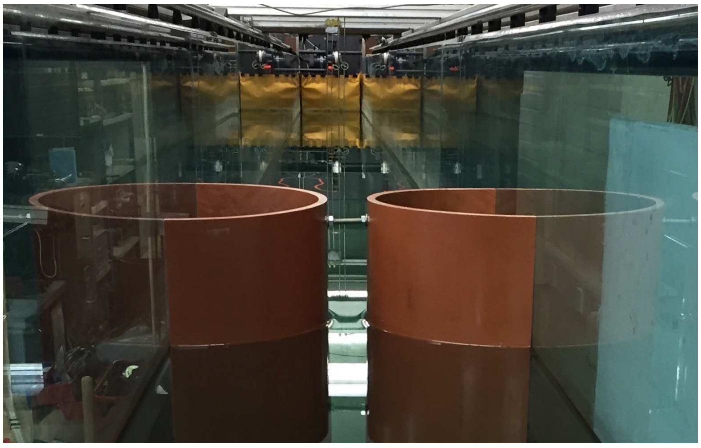

Figure 2: The wave flume, width 430mm with two half-cylinders separated by 53mm corresponding to $a/d = 0.8844$.

required to solve $\phi^{s,a}(x,y)$ in $x>0$, $y>0$ provided $\phi^a(0,y)=0$ and $\phi^s_x(0,y)=0$ with $\phi^{s,a}_y(x,0)=0$. It follows that

$$R = \tfrac{1}{2}(R^s + R^a), \qquad \text{and} \qquad T = \tfrac{1}{2}(R^s - R^a) \tag{2}$$

where $\phi^{s,a}(x,y) \to R^{s,a}\mathrm{e}^{\mathrm{i}kx}$ as $x \to \infty$ are to be determined.

A matched asymptotic method is employed, in which the quarter-domain $x>0$, $y>0$ is divided into three regimes: (i) $0 < x \ll b$, $a < y < d$; (ii) $x = O(b)$; (iii) $x \gg b$, $0 < y < d$. In (i) between the walls of the cylinder and the channel we rescale $Y = (d-y)/(d\epsilon)$ which, with $\epsilon \ll 1$, reduces (1) to $(\partial_{xx} + k^2)\phi^{s,a} = 0$ whose solutions satisfying the respective boundary conditions on $x=0$ are given by

$$\phi^s(x,y) = A^s \cos kx, \qquad \phi^a(x,y) = A^a \sin kx.$$

where $A^{s,a}$ are undetermined constants. In (iii), $x \gg b$, it is assumed the solution can be represented by an incident wave and a wave reflected by a wall spanning the full width of the channel plus a wave source located at $(x,y) = (d,b)$ to represent the effect of the small gap:

$$\phi^{s,a}(x,y) = \mathrm{e}^{-\mathrm{i}kb}(2\cos k(x-b) + m^{s,a}G(x,y))$$

where $m^{s,a}$ is an undetermined wave-source strength. Here, $G(x,y)$ can be determined explicitly in terms of separation series and from which it can be shown that

$$G(x,y) \sim \frac{-\mathrm{i}}{2kd}\mathrm{e}^{\mathrm{i}k(x-b)}, \quad kx \to \infty, \qquad \text{and} \qquad G(x,y) \sim \frac{1}{\pi}\ln(kr) + S_0(kd), \quad kr \to 0$$

where $r = ((x-b)^2 + (y-d)^2)^{1/2}$ and S_0 is a known function. In the intermediate region (ii), around the corner of the step, rescaling of coordinates $X = (x-b)/(d\epsilon)$, $Y = (d-y)/(d\epsilon)$, $\epsilon \ll 1$ maps (1) into Laplace's equation (assuming $\epsilon kd \ll 1$) corresponding to potential flow around a right-angled corner at $(X,Y) = (0,1)$ above a wall along $Y=0$. Values of $z = X + \mathrm{i}Y$ in the fluid domain are mapped, under a Schwarz-Christoffel transformation, into the upper-half ζ-plane with

$$z = \frac{2}{\pi}(1-\zeta)^{1/2} + \frac{1}{\pi}\ln\left(\frac{(1-\zeta)^{1/2}-1}{(1-\zeta)^{1/2}+1}\right)$$

in which the solution is found to be $W(\zeta) = (m^{s,a}/\pi)\ln|\zeta| + C^{s,a}$ where $C^{s,a}$ are constants to be determined by the matching.

The asymptotic matching across each of the three domains requires some algebra which eventually results in the expressions

$$R^{s,a} = \mathrm{e}^{2\mathrm{i}kb}\,\frac{2\mathrm{i}kdD^{s,a} - 1}{2\mathrm{i}kdD^{s,a} + 1} \tag{3}$$

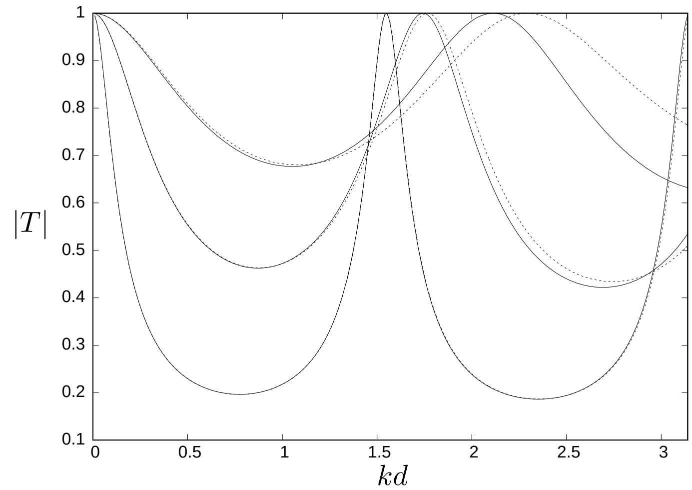

Figure 3: Variation of $|T|$ against kd for square cross-section cylinders ($b = a$) and $a/d = 0.6$, $a/d = 0.75$ and $a/d = 0.9$. Solid lines are 'exact' results and dotted lines the approximation based on matched asymptotic expansions.

with

$$D^{s,a} = \gamma + \left\{ \begin{array}{c} \cot(kb) \\ -\tan(kb) \end{array} \right\} /(2kd\epsilon), \qquad \text{where} \qquad \gamma = \frac{\ln(4\epsilon) - 1}{\pi} - \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n^2\pi^2 - (kd)^2}} - \frac{1}{n\pi} \right) \quad (4)$$

are real. Thus, $|R^{s,a}| = 1$, as required. The condition that $R = 0$ requires $R^s + R^a = 0$ and this requires the real condition $4(kd)^2 D^a D^s = -1$ be met. This condition can be expressed as

$$\cot(2kb) = \frac{kd\epsilon}{\gamma} \left(\frac{1}{4(kd\epsilon)^2} - \frac{1}{4(kd)^2} - \gamma^2 \right) \quad (5)$$

and since the right-hand side is independent of b it follows that there exist values of b such that $R = 0$, irrespective of the size of ϵ.

Fig. 3 shows curves of $|T|$ against kd below the first channel cut-off for cylinders with square cross section ($b = a$) and compares results from using the approximation (2), (3) and (4) based on matched asymptotic expansions with results computed from an exact treatment of the problem based on integral equations. We see that the agreement improves as $\epsilon \to 0$ and that zeros of reflection are manifested as increasingly rapid deviations from an underlying trend of $|T| \to 0$ as the gap decreases in size.

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