Robust computation of steady water waves of arbitrary length

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Highlights: Steady gravity waves, Arbitrary depth, Arbitrary length, Accurate computations, Fast computations.

1 Introduction

Gravity waves with a characteristic wavelength significantly larger than the mean water depth are often encountered in coastal engineering problems. Several accurate algorithms exist for computing (fully nonlinear) steady surface waves. However, none of these algorithms can compute long (cnoidal) waves, as they fail when the length-over-depth ratio exceed $30-60$, say, depending on the method. Therefore, for long waves, one has to rely on shallow water approximations. However, these series are divergent and their accuracy is limited. Moreover, their calculation is not at all trivial and numerical difficulties appear for very long waves.

Here, we present an efficient algorithm for computing the steady solutions of the Euler equation. The method works efficiently for arbitrary depth (infinite as well as shallow) and for wave heights up to about 99% of the maximum one. With this algorithm, we show that solving the Euler equations is not more demanding than the resolution of toy models, such as KdV. This claim is supported by numerical evidences.

2 Equations

We consider steady two-dimensional potential flows due to surface gravity waves in constant depth d. The fluid is of (positive) constant density ρ, the pressure is zero at the impermeable free surface, while the seabed is fixed, horizontal and impermeable. Let be (x, y) a Cartesian coordinate system moving with the wave, x being the horizontal coordinate and y being the upward vertical one. The wave is $(2\pi/k)$-periodic and $x = 0$ is the abscissa of a wave crest. By $y = -d$, $y = \eta(x)$ and $y = 0$ we denote, respectively, the equations of the bottom, of the free surface and of the mean water level. The latter implies that $\langle\eta\rangle = 0$ — $\langle\bullet\rangle$ the Eulerian average operator over one spatial period (wavelength) — i.e.

$$\langle\,\eta\,\rangle \;\stackrel{\text{def}}{=}\; \frac{k}{2\,\pi}\int_{-\pi/k}^{\pi/k}\eta(x)\,\mathrm{d}x \;=\; 0. \tag{2.1}$$

$a \stackrel{\text{def}}{=} \eta(0)$ denotes the wave crest amplitude and $b \stackrel{\text{def}}{=} -\eta(\pi/k)$ is the wave trough amplitude, so that $H \stackrel{\text{def}}{=} a + b$ is the total wave height. A wave steepness ε is then classically defined as $\varepsilon \stackrel{\text{def}}{=} kH/2$. ϕ denoting the velocity potential, the classical equations of motion are

$$\phi_{xx} \;+\; \phi_{yy} \;=\; 0 \qquad \text{for} \quad -d \leqslant y \leqslant \eta(x), \tag{2.2}$$

$$\phi_y \;=\; 0 \qquad \text{at} \qquad y \;=\; -d, \tag{2.3}$$

$$\phi_y \;-\; \eta_x\,\phi_x \;=\; 0 \qquad \text{at} \qquad y \;=\; \eta(x), \tag{2.4}$$

$$2\,g\,\eta \;+\; \phi_x^2 \;+\; \phi_y^2 \;=\; B \qquad \text{at} \qquad y \;=\; \eta(x), \tag{2.5}$$

where $g > 0$ is the (constant) acceleration due to gravity and B is a Bernoulli constant. Let be ψ, u and v the stream function, the horizontal and vertical velocities, respectively, such that $u = \phi_x = \psi_y$ and $v = \phi_y = -\psi_x$.

Consider the complex potential $f \stackrel{\text{def}}{=} \phi+\mathrm{i}\psi$ (with $\mathrm{i}^2=-1$) and the complex velocity $w \stackrel{\text{def}}{=} u-\mathrm{i}v$ that are holomorphic functions of $z \stackrel{\text{def}}{=} x+\mathrm{i}y$ (i.e., $f=f(z)$ and $w=\mathrm{d}f/\mathrm{d}z$). Using $\alpha+\mathrm{i}\beta \stackrel{\text{def}}{=} -f/c$ as independant variables, the unknown fluid domain is conformly mapped into the strip $-d \leqslant \beta \leqslant 0$. Here, c is the phase velocity observed in a frame of reference where the mean flow is zero.

After some algebra, one obtains the Babenko equation

$$B\,g^{-1}\,\tilde{y} \;-\; \tfrac{1}{2}\,\tilde{y}^2 \;+\; K_1 \;=\; \mathscr{C}^{-1}\{\tilde{y}\} \;+\; \mathscr{C}^{-1}\{\tilde{y}\,\mathscr{C}\{\tilde{y}\}\}\,, \tag{2.6}$$

where K_1 is a constant, $\tilde{y}(\alpha)$ is the equation of the free surface in the parametric conformal plane and $\mathscr{C} \stackrel{\text{def}}{=} \partial_\alpha \cot[d\partial_\alpha\,]$ is a pseudo-differential operator acting on a pure frequency as

$$\mathscr{C}\{\mathrm{e}^{\mathrm{i}\kappa\alpha}\} \;=\; \begin{cases} \kappa\coth(\kappa d)\,\mathrm{e}^{\mathrm{i}\kappa\alpha} & (\kappa\neq 0), \\ 1\,/\,d & (\kappa = 0). \end{cases} \tag{2.7}$$

The derivation of the Babenko equation (2.6) is too lengthly to be given here. Readers interested in this derivation should refer to [3] for details. In the present abstract, we focus more on the numerical method and results.

3 Petviashvili's method

Let be a nonlinear equation written $\mathcal{L}\{u\}=\mathcal{N}\{u\}$ for an unknown function $u(x)$ and where $\mathcal{L}$ is linear operator and $\mathcal{N}$ a nonlinear one. Fixed-point iterations $u_{n+1}=\mathcal{L}^{-1}\mathcal{N}\{u_n\}$ do not converge, in general. Thus, Petviashvili [6] proposed the stabilised fixed-point iterations

$$u_{n+1} \;=\; S_n^{\gamma}\,\mathcal{L}^{-1}\mathcal{N}\{u_n\}, \qquad S_n \;\stackrel{\text{def}}{=}\; \frac{\int u_n\,\mathscr{L}\{u_n\}\,\mathrm{d}x}{\int u_n\,\mathcal{N}\{u_n\}\,\mathrm{d}x}, \tag{3.1}$$

where γ is a free parameter. $\gamma=2$ is an optimum choice when $\mathcal{N}$ is quadratic in nonlinearity [5, 6].

Petviashvili's iterations are easy to implement and require $O(N)$ operations (N the number of unknowns). Conversely, Newton and Levenberg–Marquardt methods require at least $O(N^2)$ operations in theory, but $O(N^3)$ in practice for their robust implementations. However, Petviashvili's method works on a smaller class of equations, typically equations with homogeneous nonlinearities, and most often for solitary waves. It turns out that the Petviashvili method fails dramatically for computing the periodic solutions of (2.6).

In order to apply the Petviashvili method to solve the Babenko equation, we apply two simple tricks. First, instead of dealing with $\tilde{y}$, we consider the dependent variable $\Upsilon \stackrel{\text{def}}{=} \tilde{y}-\min(\tilde{y})$ so, on one period, the wave looks somehow like a solitary wave (i.e., it has a finite mass). Second, we apply the operator $\mathscr{C}_\infty \stackrel{\text{def}}{=} \mathscr{C}(d{=}\infty)$ to (2.6), thus killing the constant K_1, rendering the equation homogeneous.

The resulting Babenko equation is solvable with the Petviashvili method, for arbitrary depth, arbitrary length and for height up to about 99% of the maximum one. Matlab implementation of the algorithm can be freely downloaded [2], so interested researchers can freely use it.

4 Numerical examples

In deep water ($d\to\infty$), periodic waves with identical crests are obtained with our algorithm provided that $\varepsilon \lesssim 0.44$. Thus, the algorithm converges for rather steep waves (up to about 99.3% of the highest waves), the maximum steepness being $\varepsilon \approx 0.443164$ [4]. For $\varepsilon \leqslant 0.44$ the algorithm converges rapidly to the solution. Actually, any arbitrary accuracy can be achieved provided that N is large enough (Figure 1). For instance, for $\varepsilon=0.4$, the solution is obtained to machine double precision with $N=512$ (Fig. 1 upper). $N=1024$ is not sufficient to achieve machine quadruple precision (Fig. 1 middle), the latter being obtained for $N=2048$ (Fig. 1 lower). However, $N=2048$ is not sufficient to achieve full octuple precision, that can be obtained with larger N. Similarly, any accuracy can be obtained provided that N is large enough.

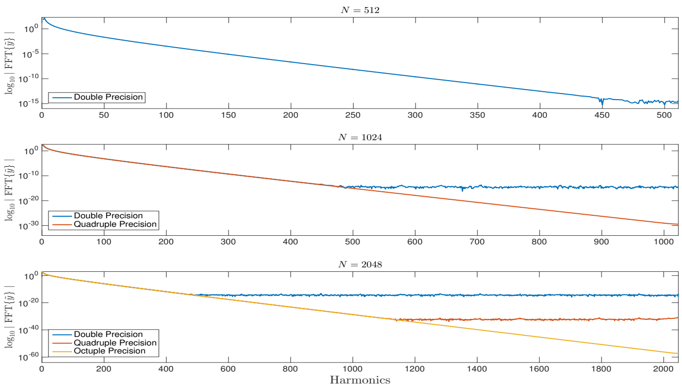

Figure 1: *Decay of the Fourier coefficients for $\varepsilon = 0.4$ and $kd = \infty$.*

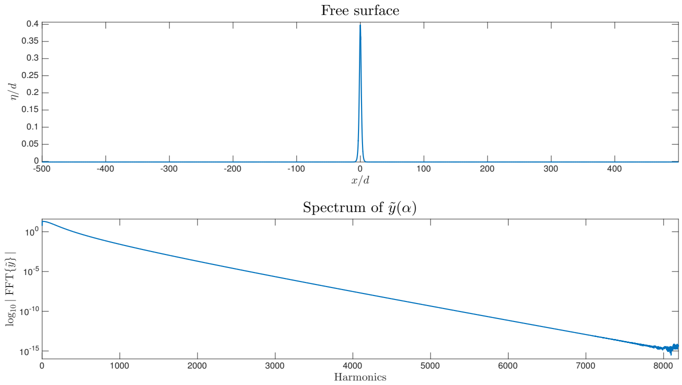

Figure 2: *Cnoidal wave in very shallow water. Upper: free surface; Lower: Fourier coefficients.*

The present algorithm works in shallow water without difficulties. For instance, with $L/d = 1000$ and $H/d = 0.4$ (i.e., $\varepsilon \approx 0.00126$ and $kd \approx 0.00628$) the solution is obtained using $N = 8192$ Fourier modes that are necessary to achieve machine double precision (Figure 2).

5 Comparison with KdV cnoidal wave

KdV analytic solution for cnoidal waves can be conveniently written

$$\eta \;=\; a\,\frac{K\,\mathrm{dn}^2(\kappa x|m)\,-\,E}{K\,-\,E}, \qquad k \;=\; \frac{\pi\,\kappa}{K}, \qquad H \;=\; \frac{m\,K\,a}{K\,-\,E}, \qquad (\kappa d)^2 \;=\; \frac{3\,H}{4\,m\,d}, \tag{5.1}$$

dn being the elliptic functions of Jacobi of parameter m ($0 \leqslant m \leqslant 1$), $K \stackrel{\text{def}}{=} \mathrm{K}(m)$ and $E \stackrel{\text{def}}{=} \mathrm{E}(m)$ being the complete elliptic integrals of the first and second kinds, respectively [1]. Tough an

L/d	100	1 000	10 000	100 000	1 000 000
$1-m$	2.04×10^{-11}	1.85×10^{-118}	6.91×10^{-1189}	3.63×10^{-11893}	5.78×10^{-118936}

Table 1: KdV parameter m for $H/d = 0.1$.

analytic expression, KdV cnoidal wave requires significant computations. Indeed, a wave being generally defined for given height H and wavenumber k, the parameter m must be determined solving numerically the equations in (5.1) relating the parameters. For very long waves, m is very

L/d	50	60	70	80	90	100
$1-m$	8.06×10^{-13}	1.77×10^{-15}	3.87×10^{-18}	8.47×10^{-21}	1.86×10^{-23}	4.06×10^{-26}

Table 2: KdV parameter m for $H/d = 0.5$.

close to one, to an extend that it cannot be practically computed (Table 1). For instance, for the very long small amplitude cnoidal wave with $L/d = 10^6$ and $H/d = 10^{-1}$ we have $1 - m \approx 5.78 \times 10^{-118936}$, a value that cannot be easily computed. This problem becomes more severe as the amplitude increases (Table 2). Thus, for very long waves, KdV analytic cnoidal solution is useless for practical applications, even if only crude approximations are sufficient. It is then more efficient to solve numerically the KdV equation, for instance with FFT and Petviashvili's method as illustrated here for the Babenko equation. But, doing so, solving KdV is not much less demanding than solving Babenko, so the latter should be preferred. For the extreme example $H/d = 10^{-1}$ and $L/d = 10^6$, Babenko solution is computed to double-precision spectral precision with $N = 2^{21}$ in about 100 s (on a MacBook Pro laptop) using our algorithm, while KdV analytic solution cannot be computed in double precision. Of course, this drawback is not limited to the KdV analytic solution, it is *a fortiori* present in all cnoidal-like approximations, such as the solutions of the Boussinesq-like equations. These considerations demonstrates that our algorithm for solving the irrotational Euler equations is also a suitable alternative to simple analytic models.

6 Conclusion

We have described an efficient algorithm for computing steady surface waves solution of the irrotational Euler equations. The algorithm is fast, accurate and it works for arbitrary wavelength and arbitrary depth. Computer wise, the algorithm being not more demanding than simple model such as KdV or high-order Stokes approximations, it is a suitable alternative to these simple models. The code is open-source [2] and can be freely used.

Other examples and improvements will be presented at the conference.

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