Head-on collision between two hydroelastic solitary waves under a thin ice sheet floating on shallow water

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Head-on collision between two hydroelastic solitary waves propagating at the surface of an incompressible and ideal fluid covered by a thin ice sheet is analytically studied by means of a singular perturbation method. The ice-sheet is modelled with the help of the special Cosserat theory of hyperelastic shells and the Kirchhoffs–Love plate theory, which yields the nonlinear and conservative expression for the bending forces. The shallow water assumption is taken. The resulting governing equations are solved asymptotically with the aid of the Poincaré–Lighthill–Kuo (PLK) method, and the solutions up to the third order are presented.

Keywords: Head-on collision, Solitary waves, hydroelasticity, surface tension, PLK method

1. INTRODUCTION

The collision of waves is also one of the major topics in marine engineering and oceanography. When the waves strike the sea coast collision occurs which creates a healthy influence on the hydrodynamics attitude of ship wakes and also on the structural attitude of the offshore structure. Scott Russell described, in 1834, the solitary waves experimentally then Korteweg and de Vries (KdV) formulated in 1895 the solitary wave mathematically in terms of the so-called KdV equation. Gardner et al. [1] described the engrossing behavior of collision between solitary gravity waves by means of the inverse scattering transform (IST) method, and found that during the collision process, the solitary waves exchange their positions and energies with each other and regain their original forms after separation.

The purpose of this paper is to examine the head-on collision between nonlinear hydroelastic solitary waves traveling in fluid covered by a thin ice sheet. The new model recently developed by Plotnikov and Toland [2] will be used for the ice sheet. For a general case, the surface tension of the fluid is also included, which is a totally new model, to the authors' best knowledge, for the fluid–ice interaction. We consider that both the solitary waves are small in amplitude ($a/H \ll 1$) having long wavelength ($\lambda/H \gg 1$), where a, H and λ are the wave amplitude, the water depth and the wave length, respectively. The physical parameters and amplitude of wavelength are related to Ursell's ordinary theory of shallow water i.e $H^3 \approx a\lambda^2$. The asymptotic solutions of the governing nonlinear equation have been obtained with the help of a singular perturbation technique named after the Poincaré–Lighthill–Kuo (PLK) method.

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2. MATHEMATICAL FORMULATION

We consider hydroelastic waves in a channel of finite depth in a Cartesian coordinate system. The horizontal plane bottom is located at $z = 0$ where the normal velocity is zero since no fluid particles penetrate the bottom. The deflection of the ice plate (namely the hydroelastic wave profile) is presented at $z = H(x, t)$ where t is the time. The velocity field for the governing flow is described by a potential function $\phi(x, z, t)$ with $\nabla^2\phi = 0$ for $0 < z < H$ and $\partial\phi/\partial z = 0$ at $z = 0$.

The kinematic boundary condition for the governing flow at the fluid–ice interface ($z = H(x, t)$) can be written as

$$\frac{\partial H}{\partial t} + \nabla\phi \cdot \nabla H = \frac{\partial \phi}{\partial z}. \tag{1}$$

The dynamic boundary condition for the floating ice sheet at the free surface is described as

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gH + \frac{P_\mathrm{e}}{\rho} = B, \tag{2}$$

where g is the gravitational acceleration, ρ the density of the fluid, P_e the pressure at the fluid–ice interface, and $B(t)$ Bernoulli's constant which, without loss of generality, will be set as $B = 0$ hereinafter. For Plotnikov and Toland's model [2], P_e takes the form of

$$P_\mathrm{e} = -T\kappa + D\left(\frac{\partial^2\kappa}{\partial s^2} + \frac{1}{2}\kappa^3\right), \tag{3}$$

where T is the coefficient of surface tension of the fluid, $D = Ed^3/[12(1-\nu^2)]$ with Young's modulus E, the thickness d and Poisson's ratio ν of the plate, respectively; κ the curvature of the fluid–ice interface, and s the arc length of this interface. The curvature κ in terms of $H(x, t)$ can be written as

$$\kappa = \frac{\partial^2 H}{\partial x^2}\left[1 + \left(\frac{\partial H}{\partial x}\right)^2\right]^{-3/2}. \tag{4}$$

According to Guyenne and Părău [3], we have

$$\frac{\partial \kappa}{\partial s} = \left[1 + \left(\frac{\partial H}{\partial x}\right)^2\right]^{-1/2} \frac{\partial \kappa}{\partial x}. \tag{5}$$

Equation (3) includes the combined effects of the surface tension of the fluid and the elasticity of the ice sheet. The second term in Eq. (3) represents Plotnikov and Toland's model, of which the linear part is the well-known linear Euler–Bernoulli beam or Kirchoff–Love plate model.

For long waves in shallow water, the potential function $\phi(x, z, t)$ can be presented as the Taylor series at $z = 0$. With the help of $\nabla^2\phi = 0$ for $0 < z < H$ and $\partial\phi/\partial z = 0$ at $z = 0$, we get

$$\phi(x, z, t) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \nabla^{2n}\Phi, \tag{6}$$

where $\Phi(x, t) = \phi(x, 0, t)$. Rewriting Eqs. (1) and (2) in terms of Φ, we get

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[HU + \sum_{n=1}^{\infty} \frac{H^{2n+1}}{(2n+1)!} \frac{\partial^{2n} U}{\partial x^{2n}}\right] = 0, \tag{7}$$

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left[gH + \frac{U^2}{2} + \frac{P_\mathrm{e}}{\rho} + \sum_{n=1}^{\infty} (-1)^n \frac{H^{2n}}{(2n)!} \left(\frac{\partial^{2n} U}{\partial t \partial x^{2n-1}} + \frac{1}{2} \sum_{m=0}^{2n} (-1)^m \mathbb{C}_m^{2n} \frac{\partial^n U}{\partial x^n} \frac{\partial^{2n-m} U}{\partial x^{2n-m}}\right)\right] = 0. \tag{8}$$

where $U = \Phi_x$ is the tangential velocity at the bottom of the channel and $\mathbb{C}_m^{2n}$ is a binomial coefficients.

3. METHOD OF SOLUTION

The solution of Eqs. (7) and (8) will be found with the help of a singular perturbation method. For this purpose, in the wave frame of reference we introduce the following coordinate transformations

$$\xi_0 = \varepsilon^{\frac{1}{2}} k(x - C_+ t), \qquad \eta_0 = \varepsilon^{\frac{1}{2}} \bar{k}(x + C_- t), \tag{9}$$

where k and $\bar{k}$ are the wave numbers of order unity for the right- and left-going wave, respectively, ε with $0 < \varepsilon \ll 1$ is a dimensionless parameter which represents the wave amplitude and order of magnitude. In accordance with Ursell's relationship the scaling of the horizontal wavelength is considered as $\varepsilon^{\frac{1}{2}}$. C_+ and C_- are the right- and left-going wave celerities, respectvively. Using the method of strained coordinates, we introduce the following transformations

$$\xi = \xi_0 + \varepsilon k \theta(\xi, \eta), \qquad \eta = \eta_0 + \varepsilon \bar{k} \varphi(\xi, \eta), \tag{10}$$

where ξ and η represent the right- and left-going phase variables; $\theta(\xi, \eta)$ and $\varphi(\xi, \eta)$ are the phase functions to be deduced in the perturbation analysis of Eqs. (7) and (8).

Let $H = H_0(1 + \zeta)$, while ζ is the nondimensional elevation of the fluid–ice interface and H_0 is the undisturbed depth of the fluid. Let $C = \sqrt{gH_0}$ be the phase speed of linear waves in shallow water of constant depth. Let us make the following changes in the dependent variables

$$U + C\zeta = 2\varepsilon C\alpha, \qquad U - C\zeta = -2\varepsilon C\beta. \tag{11}$$

Introducing the new variables in the form of following power series:

$$\alpha(\xi, \eta) = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \ldots, \tag{12}$$

$$\beta(\xi, \eta) = \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \ldots, \tag{13}$$

$$\theta(\xi, \eta) = \theta_0(\eta) + \varepsilon\theta_1(\xi, \eta) + \varepsilon^2\theta_2(\xi, \eta) + \ldots, \tag{14}$$

$$\varphi(\xi, \eta) = \varphi_0(\xi) + \varepsilon\varphi_1(\xi, \eta) + \varepsilon^2\varphi_2(\xi, \eta) + \ldots, \tag{15}$$

$$C_+ = C\left(1 + \varepsilon a R_1 + \varepsilon^2 a^2 R_2 + \ldots\right), \tag{16}$$

$$C_- = C\left(1 + \varepsilon b L_1 + \varepsilon^2 b^2 L_2 + \ldots\right), \tag{17}$$

where R_1, R_2, R_3, … and L_1, L_2, L_3 … are the coefficients for removing secular terms in the perturbation solution.

4. PERTURBATION ANALYSIS

Substituting Eqs. (12) to (17) into Eqs. (7) and (8), we get the following system of equations. The coefficients of ε, ε^2, ε^3, … are presented in sequence as follows.

4.1. Coefficients of ε

The first order solution reads as

$$\alpha_0 = aA(\xi), \qquad \beta_0 = bB(\eta), \tag{18}$$

where $A(\xi)$ and $B(\eta)$ are arbitrary functions to be determined, a and b the amplitude factors which permit us to take $A(0) = B(0) = 1$.

4.2. coefficients of ε^2

For the leading order, we categorized this section into three parts, namely secular terms, nonlocal and local terms, which will be analyzed in details as follows.

4.2.1. Secular terms. In the second order, those terms which are independent of η are the secular terms. After integrating these terms with respect η we get the secular attitude. These terms become unbounded with respect to time or space. After setting these terms equal to zero, we obtain

$$(-2R_1 + 3A)aA' + k^2 H_0^2 \left(\frac{1}{3} - \tau\right) A''' = 0, \tag{19}$$

where $\tau = T/(\rho g H_0^2)$ is a nondimensional parameter representing the effect of the surface tension of fluid. Let $R_1 = \frac{1}{2}$, $kH_0 = \sqrt{3a}$. After some simplification, Eq. (19) can be written as

$$\gamma A''' + 3AA' - A' = 0, \tag{20}$$

where $\gamma = 1 - 3\tau$. The solution of the above equation can be written as

$$A(\xi) = \mathrm{sech}^2\left(\frac{\xi}{2\sqrt{\gamma}}\right). \tag{21}$$

Similarly, we can also obtain β using the similar procedure. As τ tends to zero ($\gamma \to 1$), Eq. (21) reduces to those obtained by Su and Mirie [4] for pure gravity waves.

4.2.2. Non-local terms. These terms do not represent any secularity. So we will leave them as they are. Due to them, the resulting equation for α_1 comes under an integral. we obtain the equation for θ_0 and φ_0 as

$$\theta_0(\eta) = \frac{b}{4\bar{k}} \int_{-\infty}^{\eta} B(\eta_1)\mathrm{d}\eta_1, \tag{22}$$

$$\varphi_0(\xi) = \frac{a}{4k} \int_{+\infty}^{\xi} A(\xi_1)\mathrm{d}\xi_1. \tag{23}$$

4.2.3. Local terms. The solution for α_1 read as

$$\alpha_1(\xi,\eta) = \frac{1}{4}abAB - c_0 b^2 B + \left(\frac{3}{2}c_0 + \frac{1}{8}\right) b^2 B^2 + a^2 A_1(\xi), \tag{24}$$

where c_0 can be found using calculations. Similarly Similarly, we can also obtain β_1 using the similar procedure. In Eq. (24), $A_1(\xi)$ is an arbitrary functions will be determined in the next order of approximation.

4.3 coefficients of ε^3

Let $\Gamma = D/(\rho g H_0^4)$ is a nondimensional parameter representing the effect of the flexural rigidity of the elastic plate. The terms occurring in third order can further be summarized into three parts as follows.

4.3.1. Secular terms. The equation for the secular terms appearing in this order are

$$\gamma A_1'' + (3A - 1)A_1 = (2R_2 + c_1)A + c_2 A^2 + c_3 A^3, \tag{25}$$

where c_1, c_1 and c_3 can be obtained using calculations.

The first term on the right-hand side of Eq. (25) becomes unbounded when $\xi \to \pm\infty$, which shows that the series solution is not asymptotic. Thus, the coefficient of this term must vanish i.e.

$$R_2 = \frac{c_1}{2}. \tag{26}$$

In Eq. (25), we found that the solution for the wave celerity upto the second order are correct. The analytical solution for the rest of the terms can explicitly be written as

$$A_1(\xi) = \left(\frac{2}{3}c_2 + c_3\right) A - \frac{c_3}{2}A^2. \tag{27}$$

This completes the solution for Eq. (24). The homogeneous solution of Eq. (25) is A', but we will drop this term here because we found that when move to a higher order then the homogeneous term only causes a uniform shift of the origin of ξ which describes a simple phase shift as mentioned in the preceding section.

4.3.2. Non-local terms. The non-Local terms will provide the solution for θ_1 and φ_1 as

$$\theta_1(\xi,\eta) = \frac{b}{16\bar{k}} \int_{-\infty}^{\eta} \theta_{1,0} B \mathrm{d}\eta_1 + \frac{ab}{16\bar{k}} \int_{-\infty}^{\eta} \theta_{1,1} AB \mathrm{d}\eta_1, \tag{28}$$

where $\theta_{1,0}, \theta_{1,1}$ can be found using calculations. Similarly, we can also obtain φ_1 using the similar procedure.

4.3.3. Local terms The solution for the local terms can be written as

$$\alpha_2 = (c_4 b^2 + c_5 a^2)bB^3 + [(c_6 bB + c_7 aA)a + c_8 ab + c_9 a^2]bAB + (c_{10} b^2 + c_{11} ab - c_5 a^2)bB^2 + (c_{12} b^2 + c_{13} a^2)bB + a^3 A_2(\xi), \tag{29}$$

where $A_2(\xi)$ is arbitrary functions which can easily be found in the next order approximation using the same procedure as mentioned above. The constants c_n ($n = 4, 5, \cdots 13$) appearing in the above equations can be found using calculations. Similarly, we can also obtain β_2 using the similar procedure.

For our analysis convenience, further calculation stops at the order of $O(\varepsilon^3)$.

5. SOLUTION OF THE PROBLEM

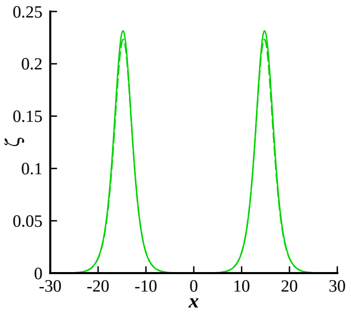
The major results are obtained in preceding section are deduced as follows.

The surface elevation at the water–plate interface can be obtained from Eq. (11). Thus we have

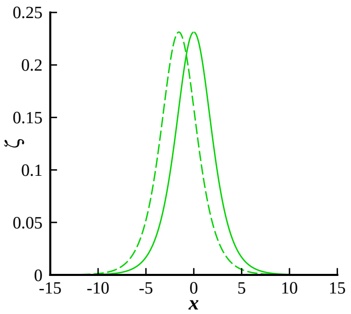
$$\zeta = \varepsilon(\alpha + \beta). \tag{30}$$

The distortion profile can be obtained. After setting $B(\eta) = 0$, we get

$$\zeta = \varepsilon a A + \varepsilon^2 a^2 \left[\frac{1}{2}\left(c_0 + \frac{1}{4}\right) A + A_1(\xi)\right] + O(\varepsilon^3), \tag{31}$$



(a)



(b)

Fig. 1. (a) Head-on collision between two solitary waves. Solid line: $\Gamma = 0$; Dash line: $\Gamma = 0.007$. (b) Distortion profile. Solid line: Before collision; Dashed line: After collision.

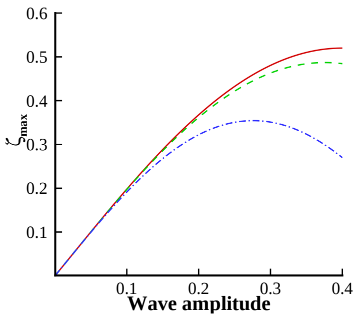


Fig. 2. Maximum run-up *vs* wave amplitude. Solid line: $\Gamma = 0$; Dashed line: $\Gamma = 0.007$; Dash-dot line: $\Gamma = 0.03$.

The maximum run-up $\zeta_{\max}$ during the collision process can be obtained by taking $A = B = 1$ in Eq. (31), we get $\zeta_{\max} = \zeta\Big|_{A=B=1}$.

The velocity at the bottom U can be obtained from Eq. (11), we get

$$U/C = \varepsilon(\alpha - \beta). \tag{32}$$

Following from Eqs. (16) and (17), the asymptotic solutions for the wave speeds read

$$C_+/C = 1 + \frac{1}{2}\varepsilon a - \frac{c_1}{2}\varepsilon^2 a^2 + O(\varepsilon^3), \tag{33}$$

$$C_-/C = 1 + \frac{1}{2}\varepsilon b - \frac{c_1}{2}\varepsilon^2 b^2 + O(\varepsilon^3). \tag{34}$$

The phase shifts during the collision process reads

$$\theta = \theta_0 + \varepsilon\theta_1 + O(\varepsilon^2), \tag{35}$$

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + O(\varepsilon^2), \tag{36}$$

where θ_0, θ_1, φ_0, and φ_1 are given in preceding section.

6. DISCUSSION

We describe the graphical results for all the pertinent parameters involved in this hydroelastic wave problem. For this purpose, Figs. 1 and 2 have been sketched against the important parameters. We take the physical parameters i.e. $E = 10^4\,\mathrm{N\,m^{-2}}$, $d = 0.007\,\mathrm{m}$, $g = 9.8\,\mathrm{m\,s^{-2}}$, $\rho = 1000\,\mathrm{kg\,m^{-3}}$, $H_0 = 1\,\mathrm{m}$, and $T = 0.075\,\mathrm{N\,m^{-1}}$ for the graphical results.

Figure 1(a) is plotted for multiple values of Γ. In this figure we observe that the significant enhancement in the flexural rigidity parameter Γ, providing a significant resistance. An increasing Γ tends to diminish the interfacial surface elevation. Physically, when Young's modulus E rises, the deflection of a plate becomes stiffer and a very high rate of reactive force occurs to oppose the deformation of an elastic plate. Figure 1(b) represents the distortion in the wave profile which is plotted with the help of Eq. (31). Before collision when $\theta = 0$, the wave profile is symmetric, however after collision process $\theta \neq 0$, the wave profile is not symmetric and tilts backward in the direction of propagation. It depicts in Fig. 2 that Γ tends to diminish the maximum run-up as the wave amplitude rises.

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