Stationary phase and numerical evaluation of far-field and near-field ship waves in shallow water

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1 INTRODUCTION

Numerical evaluation of the far-field and near-field waves created by a ship hull, of length L, that advances at constant speed V along a straight path in calm water of uniform finite depth D are considered within the classical framework of linear potential flow theory. This theoretical framework is realistic and actually the only practical option to calculate the far-field ship waves. As is illustrated in Zhang (2015), the authors have developed the practical method to calculate the far-field and near-field ship waves in deep water. Now the similar idea is applied to shallow water situation. The flow pressure over a ship hull surface can be formally decomposed into a non-oscillatory local-flow component and an oscillatory wave component. The wave component yields a major, indeed dominant contribution to the wave drag, the hydrodynamic lift and moment, and the related sinkage and trim experienced by a ship. Thus, numerical evaluation of the wave component is a critical element of the computation of the near-field flow around a ship hull, as well as far-field ship waves as already shown in Zhang (2015).

The Froude number F and the nondimensional water depth d^L and d^V are defined as

$$F \equiv \frac{V}{\sqrt{gL}} \equiv \sqrt{\frac{d^L}{d^V}}, \qquad d^L \equiv \frac{D}{L}, \quad d^V \equiv \frac{Dg}{V^2} \equiv \frac{d^L}{F^2}$$
(1)

where g denotes the acceleration of gravity. Finite water depth effects are only significant if $d^V < d_{\infty}^V$ with $d_{\infty}^V \approx 3$, and the water depth can then be regarded as effectively infinite for $d_{\infty}^V < d^V$. The waves and the related flow around the ship hull are observed from an orthogonal frame of reference and related coordinates $\mathbf{X} \equiv (X, Y, Z)$ attached to the ship. As is shown in Fig.1, the Z axis is vertical and points upward, with the undisturbed free surface taken as the plane Z = 0. The X axis is chosen along the path of the ship and points toward the ship bow. Far-field ship waves are then found in the region X < 0 aft of the ship. Nondimensional coordinates and the corresponding Fourier variables are defined as

$$\mathbf{x} \equiv (x, y, z) \equiv (X, Y, Z)g/V^2 \quad \text{and} \quad (\alpha, \beta, k \equiv \sqrt{\alpha^2 + \beta^2}) \equiv (A, B, K)V^2/g \tag{2}$$

where

$$(\alpha,\beta) \equiv \left(\sqrt{k \tanh(kd/F^2)}, \pm \sqrt{k^2 - k \tanh(kd/F^2)}\right)$$
(3)

As shown in Fig.1,**x** stands for a flow field point located within the flow region outside the mean wetted ship hull surface, denoted as Σ^{H} , and $\boldsymbol{\xi} \equiv (\boldsymbol{\xi}, \eta, \boldsymbol{\zeta})$ denotes a point of Σ^{H} .

Within the context of linear potential flow theory considered here, the elevation $e \equiv Zg/V^2$ of the free surface above or below the plane Z = 0 is given by $e(x, y) = \phi_x(x, y, z = 0)$. Moreover, similar with the deep water stuation, the wave elevation e can be expressed as a linear superposition of elementary wave functions Edefined as

$$E \equiv e^{ih\varphi} \quad \text{with } \varphi = \alpha \cos \psi + \beta \sin \psi \tag{4}$$

Here, φ denotes the phase of the elementary wave function *E*. *h* denotes the horizontal distance between field point and the origin x = 0, y = 0, taken at the centroid of the ship water plane. ψ is the polar angle measured from the negative *x* axis (ship track), i.e.

$$h \equiv \sqrt{x^2 + y^2} \equiv \frac{Hg}{V^2} \equiv \frac{H/L}{F^2} \quad \text{and} \quad \tan \psi \equiv \frac{y}{-x}$$
(5)

Specifically, the free-surface elevation e(x, y) can be represented in terms of the Fourier integral

$$e(x,y) = \operatorname{Re} \int_{k_0}^{k_{\infty}} \Lambda AEdk \quad \text{with} \quad k_{\infty} \equiv (2+10F)^2 \tag{6}$$



Fig. 1: Definition sketch for the computation of ship waves created by a ship hull that advances at constant speed along a straight path in calm water of finite water depth D.



Fig. 2: Variations of the asymptote angle $\psi_{asymp}(d^V)$ for $0 \le d^V \le 1$ (dashed line) and of the cusp angle $\psi_{cusp}(d^V)$ for $1 \le d^V$ (solid line) with respect to $0 \le d^V \le 3$.

where k_0 is defined as

$$k_0 = \begin{cases} \text{positive root of } k = \tanh(kd^V) & \text{if } d^V > 1\\ 0 & \text{if } d^V \le 1 \end{cases}$$
(7)

the finite range of integration k_{∞} and the filter function Λ in Eq.6 are used to eliminate unrealistic short waves, as now briefly explained.

The filter function Λ in Eq.6 is based on parabolic extrapolation, as given in Huang (2013). The extrapolation height in this parabolic extrapolation filter is extended here from the ship hull surface Σ^{H} , as considered in Huang (2013), into the flow region outside Σ^{H} in the same straight-forward manner as in Zhang (2015).

The amplitude function A, also called wave-spectrum function, in the Fourier-Kochin representation (6) is defined as a superposition of elementary wave functions over the ship hull surface Σ^H . For instance, the amplitude function $A \equiv A(k, \xi, d^V)$ associated with the classical Hogner approximation is defined explicitly in terms of the Froude number F, the water depth d^V and the shape of the ship hull surface Σ^H as

$$A = \frac{2}{F^4} \int_{\Sigma^H} H(\xi - x) \frac{\cosh\left[k(z + d^V)\right]}{\cosh(kd^V)} \frac{e^{-i(\alpha\xi + \beta\eta)}}{\sqrt{1 - \tanh(kd^V)/k}} n_x(\boldsymbol{\xi}) da(\boldsymbol{\xi}) \tag{8}$$

where $H(\xi - x)$ denotes the Heaviside unit-step function, $da(\xi)$ and $n_x(\xi)$ stand for the differential element of area at a point $\xi \in \Sigma^H$ and x component of the unit vector $\mathbf{n} \equiv (n_x, n_y, n_z)$ normal to $\xi \in \Sigma^H$, repectively.

The Fourier integral (6) is a major critical element of the computation of near-field and far-field ship waves. Expressions (5) and (4) show that the trigonometric function E in the Fourier integration(6) for $1 \ll h$ therefore requires a huge number of integration points and is not practical if a typical integration rule (e.g. Gaussian or Simpson) is used.

A simple analytical approximation, based on Kelvin's classical method of stationary phase in Thomson (1887), can be used in the far field $1 \ll h$ for $|\psi| < \psi_{cusp}$ or $|\psi| < \psi_{asymp}$ if $d^V > 1$ or $d^V \leq 1$ as illustrated in Fig.2. This classical stationary-phase approximation, considered furthermore, is not valid in the vicinity of the cusp $\psi = \pm \psi_{cusp}$ or asymptote lines $\psi = \pm \psi_{asymp}$, i.e. is only valid strictly inside the cusp or the asymptote wedge. Like the far-field stationary-phase analytical approximation, the Filon numerical integration rule in Filon (1928) for rapidly oscillating integrands is not suited in the nearfield. Moreover, this integration rule is not well suited for the case when points of stationary phase are present.

2 STATIONARY PHASE APPROXIMATION

As is noted in Kelvin's theory, for large values of h, where the trigonometric function E oscillates rapidly, the dominant contributions to the integral (6) stem from points where the phase φ is stationary, or points where $\varphi' \equiv d\varphi/dk = 0$. Expression (2) and (3) for the phase function φ yields

$$\varphi' = \alpha' \cos \psi + \beta' \sin \psi \tag{9}$$

where

$$(\alpha',\beta') \equiv \left(\frac{d^V k \operatorname{sech}^2(d^V k) + \tanh(d^V k)}{2\sqrt{k \tanh(d^V k)}}, \pm \frac{2k - d^V k \operatorname{sech}^2(d^V k) - \tanh(d^V k)}{2\sqrt{k(k - \tanh(d^V k))}}\right)$$
(10)



Fig. 3: Variations of the phase function's derivative $\varphi'(k, \psi)$ within the range $0 \le k \le 30$ for four ray angles $\psi = 5^{\circ}, \psi = 20^{\circ}, \psi = 35^{\circ}$ and $\psi = 55^{\circ}$ in two water depth $d^{V} = 0.625$ (left) and $d^{V} = 1.25$ (right).

Fig.3 illustrates the derivative of phase function φ' within the range $0 \le k \le 30$ for four ray angles $\psi = 5^{\circ}$, $\psi = 20^{\circ}$, $\psi = 35^{\circ}$ and $\psi = 55^{\circ}$, in two water depth $d^{V} = 0.625$ (left) and $d^{V} = 1.25$ (right).

As we can see for $d^V \leq 1$ and $\psi \leq \psi_{asymp}$ there is one root k^A for $\varphi' = 0$, and under this circumstances only divergence waves can be found in the wave pattern generated by a ship. As for $\psi > \psi_{asymp}$ there is no root for $\varphi' = 0$ and φ' takes to the minimum at k = 0. So ideally, there should be no waves in the very far field out of the wake angle ψ_{asymp} . In the case of $d^V > 1$, when $\psi < \psi_{cusp}$ there are two roots k^T and k^D , which correspond to the transverse and divergent waves respectively, for $\varphi' = 0$. And when $\psi = \psi_{cusp}$, there is only one root k^C for $\varphi' = 0$, or we can say $k^T = k^D$ where wavelength of transverse and divergent waves equal. As for $\psi > \psi_{cusp}$, there is no root for $\varphi' = 0$ and φ' takes to the minimum at k^C where $\varphi'' = 0$. As we can see, the conclusions for $d^V > 1$ are similar to the case in deep water and the ψ_{cusp} approaches the Kelvin angle $\psi^K \approx 19^\circ 28'$ with inscreasing of d^V . Generally, as mensioned before, we regard it as deep water when $d^V > 3$.

3 IMPLEMENTATION AND CONCLUSION

Considering the fact that the dominant contributions of the Fourier integral (6) stem from points of stationary phase, where $\varphi' = 0$. The trigonometric function E, defined by (4), can then be modified as

$$\widetilde{E} \equiv e^{ih\varphi - (h\Delta\varphi/2n\pi)^4} \tag{11}$$

with

$$\Delta \varphi \equiv \begin{cases} |\varphi(k) - \varphi(k^{A})| & \text{if } d^{V} < 1 \text{ and } \psi < \psi_{asymp} \\ |\varphi(k) - \varphi(0)| & \text{if } d^{V} < 1 \text{ and } \psi \ge \psi_{asymp} \\ \min(|\varphi(k) - \varphi(k^{T})|, |\varphi(k) - \varphi(k^{D})|) & \text{if } d^{V} > 1 \text{ and } \psi < \psi_{cusp} \\ |\varphi(k) - \varphi(k^{C})| & \text{if } d^{V} > 1 \text{ and } \psi \ge \psi_{cusp} \end{cases}$$
(12)

where k^A, k^T, k^D, k^C, d^V are defined hereinbefore and $\psi_{asymp}, \psi_{cusp}$ are illustrated in Fig.2. *n* determined the affected domain of the modified term. Specifically, $1\% \leq \tilde{E}/E \leq 99\%$ when $h\Delta\varphi$ varies from $0.63n\pi$ to $2.92n\pi$. Within the range $0 < h\Delta\varphi < 0.63n\pi$, the modified term rarely affect the trigonometric elementary wave function *E*. As a result, $\tilde{E} \approx E$ in the vicinity of the stationary phase points and with a proper value of *n*, the numerical calculation accuracy of the Fourier intergral (6) can be well guaranteed. And inside the band of $0.63n\pi \leq h\Delta\varphi \leq 2.92n\pi$, the modified function \tilde{E} is smoothly decayed from function *E* to 0. It is obvious that if a larger power number of the modified term $h\Delta\varphi/2n\pi$ is chosen, instead of 4 now, the band will become narrower. here for simplicity we just take 4 as an example to illustrate the main spirit of our method. When $h\Delta\varphi$ is greater than $2.92n\pi$, oscillations of the trigonometric function *E* are almost damped. As we know, in the far field where $h \to \infty$ the function *E* oscillates rapidly, thus accurate numerical evaluation of the Fourier integral (6) requires a huge number of integration points, which may yield badly low computing efficiency. So with the help of the modified function (11), which damps the oscillatory part of the function *E*, the Fourier integral (6) can be calculated efficiently and even more accurately.

As we can see, the representation of the modified trigonometric function \tilde{E} is quite simple that only one parameter *n* need determined. The modified trigonometric function $\tilde{E} \approx E$ in the vicinity of the stationary phase points where $\varphi' = 0$ or points where $\varphi'' = 0$ or points k = 0 as is depicted in (12). And due to the existence of *h* in the term $h\Delta\varphi$, the modified function (11) is practically identical to the original function (4) in the near field. So the dominant parts, that contribute to the Fourier integral (6), in function (4) are not modified and therefore guarantee the accuracy the numerical evaluation. As for the rapidly oscillatory parts of the trigonometric function E that do not contribute to the Fourier integral (6) are damped by the modified function \tilde{E} that yield computational efficiency without negatively affecting accuracy. The effectiveness of the modified function with n = 4 are demonstrated in Fig.4 and Fig.5. Furthermore, the modified function \tilde{E} , defined by (11), for practical numerical evaluation of ship waves in finite water depth is much simpler than the representation given in Zhang (2015) for deep water. As is mentioned before, with increasing of d^V the finite water cases will develope into deep water cases. So the modification made in this paper can be also used to calculate near-field and far-field ship waves in deep water. Fig.6 illustrate ship waves of the Wigley hull calculated by equation (6) with E replaced by \tilde{E} in three water depth from shallow $d^V = 0.625$ to deep $d^V = 3$ at Froude number F = 0.3. It shows the method's applicability to calculate ship waves in near-field and far-field both for finite and infinite water depth.



Fig. 4: Real parts of the trigonometric function E defined by (4) and the related modified function \overline{E} defined by (11) for h = 5, 25, 100 and $\psi = 5^{\circ}, 20^{\circ}, 35^{\circ}, 50^{\circ}$ with $d^{V} = 0.625$



Fig. 5: Real parts of the trigonometric function E defined by (4) and the related modified function \tilde{E} defined by (11) for h = 5, 25, 100 and $\psi = 5^{\circ}, 20^{\circ}, 35^{\circ}, 50^{\circ}$ with $d^{V} = 1.25$



Fig. 6: Ship waves generated by a Wigley hull at Froude number F = 0.3 in three water depth.

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