Action of Periodic External Pressure on Inhomogeneous Ice Cover

Izolda V. Sturova* and Larisa A. Tkacheva

Lavrentyev Institute of Hydrodynamics of SB RAS, Novosibirsk, Russia
E-mail: sturova@hydro.nsc.ru

HIGHLIGHT

- The analytic solution of the 3-D problem on the flexural-gravity waves generated by a local time-periodic external pressure on inhomogeneous ice cover is obtained.
- For two identical ice sheets with a rectilinear crack, the solution is received in explicit form.

1 INTRODUCTION

The behavior of the ice cover under dynamic action has been thoroughly studied for the homogeneous ice sheet, covering the water surface completely (e.g. Squire et al., 1996). In the case of inhomogeneous ice cover, there are only some examples of solutions of particular problems.

In this paper, three-dimensional flexural-gravity waves generated by a local time-periodic external pressure on top of ice sheet are investigated for three configurations, concerning: (i) a freely floating semi-infinite ice sheet, (ii) two semi-infinite ice sheets with different properties connected by rectilinear partially frozen crack, and (iii) a semi-infinite ice sheet in contact with the fixed vertical wall (the edge of the ice sheet can be either clamped or free). The problem is formulated within linear hydroelastic theory. The fluid is assumed to be inviscid and incompressible and its motion is potential. The ice sheet is treated as an elastic thin plate. The behavior of amplitudes of the plate deflection and water elevation in dependence on the frequency and the position of the load center is analyzed. The solutions are obtained both for a fluid of finite depth and for shallow-water approximation.

2 MATHEMATICAL FORMULATION

Let us consider the statement of the problem in the most complex configuration (ii). Two semi-infinite elastic plates of thicknesses h_1 and h_2 float on water of depth H. These plates may be connected at $x = 0$ by a vertical linear spring and a flexural rotational spring with stiffness k_1 and k_2, respectively. These two springs simulate a partially frozen crack. The y-axis is directed along the rectilinear crack and the z-axis is directed vertically upwards. The plate drafts are ignored. The wave motions of the fluid and the ice sheets are generated by the steady forced oscillations of the external pressure $p(x, y, t) = P(x, y) \exp(-i\omega t)$, where ω is the frequency and t is the time. We restrict our consideration to local axially symmetric load with the parabolic distribution, $P(x, y) = g\rho_0 H f(r_0)$, $f(r_0) = 1 - (r_0/a)^2$, $r_0 = \sqrt{(x - x_0)^2 + y^2}$. Here ρ_0 is the fluid density, g is the acceleration due to gravity, a is radius of the load domain, the center of which is at the point $x = x_0,\ y = 0\ (x_0 > a)$.

The boundary-value problem for the velocity potential $\varphi(x, y, z, t)$ and deflection of ice sheet $w(x, y, t)$ can be written as

$$\Delta_3\varphi = 0 \quad (-\infty < x, y < \infty,\ -H < z < 0), \quad \Delta_3 \equiv \Delta_2 + \partial^2/\partial z^2, \quad \Delta_2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2\ , \tag{1}$$

$$D_n\Delta_2^2 w + \rho h_n \partial^2 w/\partial t^2 + g\rho_0 w + \rho_0 \partial\varphi/\partial t = \mathcal{H}(x)p(x, y, t) \quad (z = 0), \tag{2}$$

$$\partial w/\partial t = \partial\varphi/\partial z\ (z = 0), \qquad \partial\varphi/\partial z = 0 \quad (z = -H), \tag{3}$$

$$D_1\Big(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\Big)w_- = k_2\Big(\frac{\partial w_+}{\partial x} - \frac{\partial w_-}{\partial x}\Big), \quad D_2\Big(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\Big)w_+ = k_2\Big(\frac{\partial w_+}{\partial x} - \frac{\partial w_-}{\partial x}\Big), \tag{4}$$

$$D_1\frac{\partial}{\partial x}\Big[\frac{\partial^2}{\partial x^2} + (2 - \nu)\frac{\partial^2}{\partial y^2}\Big]w_- = k_1(w_- - w_+), \quad D_2\frac{\partial}{\partial x}\Big[\frac{\partial^2}{\partial x^2} + (2 - \nu)\frac{\partial^2}{\partial y^2}\Big]w_+ = k_1(w_- - w_+), \tag{5}$$

where $w_- = w|_{x=-0},\ w_+ = w|_{x=+0}$. Here $D_n = Eh_n^3/[12(1 - \nu^2)]$; $E,\ \nu,\ \rho$ are the Young's modulus, the Poisson's ratio and the density of the ice sheet, respectively; $\mathcal{H}(x)$ is the Heaviside function; $n = 1$

at $x < 0$ and $n = 2$ at $x > 0$. The edge conditions (4), (5) are the most general boundary conditions for partially frozen crack. Taking the limit values for k_1 and k_2, we can also model the free-edge ($k_1 = k_2 = 0$), hinge-connector ($k_1 = \infty,\ k_2 = 0$) and rigidly joined plates ($k_1 = k_2 = \infty$) cases. The radiation condition is imposed in the far field.

For $D_1 = 0$, we have the configuration (i) in which the fluid is bounded by the free surface at $x < 0$ and the edge of the ice sheet is free. For the configuration (iii), the fluid is restricted at the left by the rigid wall: $\partial\varphi/\partial x = 0$ at $x = 0$. The edge of the ice sheet can be free or frozen to the fixed vertical structure, then $w = \partial w/\partial x = 0 \quad (x = 0)$.

For *the shallow-water approximation* we can easily take into account the draft of the plates. The potential $\varphi(x, y, t)$ and the deflection $w(x, y, t)$ are determined from the relations (2), (4), (5) and the equation $\partial w/\partial t = -(H - d_n)\Delta_2\varphi$, where $d_n = \rho h_n/\rho_0 \quad (n = 1, 2)$.

3 METHOD OF SOLUTION

We describe briefly the solution of the problem (1)∼(5) by the Wiener-Hopf technique. The dimensionless variables and parameters are introduced

$$(x', y', z') = \frac{1}{H}(x, y, z),\ \lambda = \omega\sqrt{\frac{H}{g}},\ \ \beta_n = \frac{D_n}{g\rho_0 H^4},\ \delta_n = \frac{\rho h_n}{\rho_0 H}\lambda^2,\ k_1' = \frac{k_1 H^3}{D_2},\ \ k_2' = \frac{k_2 H}{D_2}.$$

Below, the primes are omitted. We will seek the velocity potential and the deflection in the form

$$\varphi(x, y, z, t) = \sqrt{gH^3}\phi(x, y, z)\mathrm{e}^{-i\omega t},\ w(x, y, t) = HW(x, y)\mathrm{e}^{-i\omega t}.$$

We use the Fourier transform to the variables x and y in the form

$$\Phi^-(\alpha, s, z) = \int_{-\infty}^{\infty}\mathrm{e}^{-isy}\int_{-\infty}^{0}\phi(x, y, z)\mathrm{e}^{i\alpha x}dxdy, \quad \Phi^+(\alpha, s, z) = \int_{-\infty}^{\infty}\mathrm{e}^{-isy}\int_{0}^{\infty}\phi(x, y, z)\mathrm{e}^{i\alpha x}dxdy.$$

From the Laplace equation (1) and the no-flux bottom condition (3), we have

$$\Phi(\alpha, s, z) = \Phi^-(\alpha, s, z) + \Phi^+(\alpha, s, z) = C(\alpha, s)Z(\alpha, s, z), \quad Z(\alpha, s, z) = \frac{\cosh[(z+1)\sqrt{\alpha^2 + s^2}]}{\cosh\sqrt{\alpha^2 + s^2}}, \tag{6}$$

where $C(\alpha, s)$ is unknown function. We introduce the functions $G_n^{\pm}(\alpha, s)$ $(n = 1, 2)$

$$G_n^-(\alpha, s) = \int_{-\infty}^{\infty}\mathrm{e}^{-isy}\int_{-\infty}^{0}\Big[(\beta_n\Delta_2^2 + 1 - \delta_n)\frac{\partial\phi}{\partial z} - \lambda^2\phi\Big]_{z=0}\mathrm{e}^{i\alpha x}dxdy,$$

$$G_n^+(\alpha, s) = \int_{-\infty}^{\infty}\mathrm{e}^{-isy}\int_{0}^{\infty}\Big[(\beta_n\Delta_2^2 + 1 - \delta_n)\frac{\partial\phi}{\partial z} - \lambda^2\phi\Big]_{z=0}\mathrm{e}^{i\alpha x}dxdy.$$

The functions with the indexes $+/-$ are analytical on α in the upper/lower half-plane, respectively. From the boundary conditions (2), we have

$$G_1^-(\alpha, s) \equiv 0,\ \ G_2^+(\alpha, s) = -i\lambda F(\alpha, s)\mathrm{e}^{i\alpha x_0},\ \ F(\alpha, s) = 4\pi\mathrm{e}^{i\alpha x_0}J_2(a\sqrt{\alpha^2 + s^2})/(\alpha^2 + s^2), \tag{7}$$

where the $F(\alpha, s)$ is the Fourier transform of the function $f(r_0)$ and $J_2(\cdot)$ is a Bessel function of the first kind. Using (6), we have

$$G_n(\alpha, s) = G_n^-(\alpha, s) + G_n^+(\alpha, s) = C(\alpha, s)K_n(\alpha, s), \tag{8}$$

where $K_n(\alpha, s)$ are the dispersion functions for the flexural-gravity waves

$$K_n(\alpha, s) = [\beta_n(\alpha^2 + s^2)^2 + 1 - \delta_n]\sqrt{\alpha^2 + s^2}\tanh(\sqrt{\alpha^2 + s^2}) - \lambda^2 \quad (n = 1, 2).$$

It is known, that the dispersion relation $\mathcal{K}_1(\gamma) \equiv (\beta_1\gamma^4 + 1 - \delta_1)\gamma \tanh\gamma - \lambda^2 = 0$ has real roots $\pm\gamma_0$, four complex roots $\pm\gamma_{-1}$, $\pm\gamma_{-2}$, $\gamma_{-2} = -\bar{\gamma}_{-1}$ (the bar denotes complex conjugation), and the countable set of imaginary roots $\pm\gamma_m,\ m = 1, 2, \ldots$. The second relation $\mathcal{K}_2(\mu) \equiv (\beta_2\mu^4 + 1 - \delta_2)\mu \mathrm{th}\mu - \lambda^2 = 0$ has the roots μ_k $(k = -2, -1, 0, \ldots)$. Then the roots of the dispersion relations $K_n(\alpha, s) = 0$ are $\chi_m = \sqrt{\gamma_m^2 - s^2}$ $(n = 1)$, $\alpha_m = \sqrt{\mu_m^2 - s^2}$ $(n = 2)$.

From the relations (7) and (8), we obtain

$$G_2^-(\alpha, s) - i\lambda F(\alpha, s)\mathrm{e}^{i\alpha x_0} = G_1^+(\alpha, s)K(\alpha, s), \quad K(\alpha, s) = K_2(\alpha, s)/K_1(\alpha, s).$$

We factorize the function $K(\alpha, s)$

$$K(\alpha, s) = K^-(\alpha, s)K^+(\alpha, s), \quad K^\pm(\alpha, s) = \prod_{j=-2}^{\infty} (\alpha \pm \alpha_j)\gamma_j/[\mu_j(\alpha \pm \chi_j)],$$

where $K^\pm$ are analytical in the upper/lower parts of the complex plane α, respectively.

We use the representation by Noble [1958]

$$\frac{F(\alpha, s)\mathrm{e}^{i\alpha x_0}}{K^-(\alpha, s)} = L^-(\alpha, s) + L^+(\alpha, s), \quad L^\pm(\alpha, s) = \pm\frac{1}{2\pi i}\int_{-\infty\mp i\sigma}^{\infty\mp i\sigma} \frac{F(\zeta, s)\mathrm{e}^{i\zeta x_0} d\zeta}{K^-(\zeta, s)(\zeta - \alpha)},$$

and as a result we obtain the equation

$$G_2^-(\alpha, s)/K^-(\alpha, s) - i\lambda L^-(\alpha, s) = G_1^+(\alpha, s)K^+(\alpha, s) + i\lambda L^+(\alpha, s).$$

The functions on the left and right sides of this equation are analytical in the lower and upper parts of the complex plane α, respectively. Then we have analytical function over the entire complex plane α. By Liouville's theorem, this function is a polynomial. The degree of the polynomial is determined by the behavior of functions as $|\alpha| \to \infty$ and is equal to three:

$$G_1^+(\alpha, s)K^+(\alpha, s) + i\lambda L^+(\alpha, s) = \sum_{k=0}^{3} a_k(s)\alpha^k,$$

where $a_k(s)$ are unknown functions which are defined from the edge conditions. We have

$$\Phi(x, s, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-i\alpha x}\Big[\sum\limits_{k=0}^{3} a_k\alpha^k - i\lambda L^+(\alpha, s)\Big] Z(\alpha, s, z) d\alpha}{K^+(\alpha, s)K_1(\alpha, s)}.$$

Using the conditions (4),(5), we obtain the system of four linear algebraic equations to define the coefficients $a_k(s)$ $(k = 0, 1, 2, 3,)$. All integrals are evaluated by the residue method.

After solving this system, we find the ice deflections by performing inverse Fourier transform: for $x < 0$

$$W(x, y) = -\frac{\lambda}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{isy} \sum_{j=-2}^{\infty} \frac{\mathrm{e}^{-i\chi_j x}\Big[\sum\limits_{k=0}^{3} a_k\chi_j^k - i\lambda L^+(\chi_j, s)\Big]}{K^+(\chi_j, s)K_1'(\chi_j, s)(\beta_1\gamma_j^4 + 1 - \delta_1)} ds, \tag{9}$$

for $x > 0$

$$W(x, y) = 2\int_0^{\infty} \frac{J_0(\zeta r_0)J_2(a\zeta)\tanh\zeta d\zeta}{\mathcal{K}_2(\zeta)} -$$

$$\frac{\lambda}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{isy} \sum_{j=-2}^{\infty} \frac{\sum\limits_{k=0}^{3} a_k(-\alpha_j)^k + i\lambda L^-(-\alpha_j, s)}{K_2'(\alpha_j, s)(\beta_2\mu_j^4 + 1 - \delta_2)} \mathrm{e}^{i\alpha_j x} K^+(\alpha_j, s) ds. \tag{10}$$

The integrands in (9) and (10) decay exponentially as $s \to \infty$. The first term in (10) corresponds to the solution for the infinitely extended ice cover.

For the identical sheets with a crack, the solution can be obtained in the explicit form. In this case $K^{\pm}(\alpha, s) \equiv 1,\ L^{-}(\alpha, s) \equiv 0$. The coefficients $a_0(s),\ a_2(s)$ and $a_1(s),\ a_3(s)$ determine the symmetric and antisymmetric part of the solution, respectively. The system of linear algebraic equation for the determination of coefficients a_k $(k = 0, 1, 2, 3)$ is divided on two systems. We find $a_0(s) = \nu s^2 a_2(s),\ \ a_1(s) = (2-\nu)s^2 a_3(s)$. The solution for $a_2(s)$ and $a_3(s)$ has the form

$$a_2\left[\Lambda(s) + i2k_2 \sum_{j=-2}^{\infty} \frac{\alpha_j(\alpha_j^2 + \nu s^2)}{K_2'(\alpha_j, s)(\beta_2\mu_j^4 + 1 - \delta_2)}\right] = i4\pi\lambda \sum_{j=-2}^{\infty} \frac{(\alpha_j^2 + \nu s^2)\mathrm{e}^{i\alpha_j x_0} J_2(a\mu_j)}{\mu_j^2 K_2'(\alpha_j, s)(\beta_2\mu_j^4 + 1 - \delta_2)},$$

$$a_3\left[\Upsilon(s) + i2k_1 \sum_{j=-2}^{\infty} \frac{\alpha_j[\alpha_j^2 + (2-\nu)s^2]}{K_2'(\alpha_j, s)(\beta_2\mu_j^4 + 1 - \delta_2)}\right] = i4\pi\lambda \sum_{j=-2}^{\infty} \frac{\alpha_j[\alpha_j^2 + (2-\nu)s^2]\mathrm{e}^{i\alpha_j x_0} J_2(a\mu_j)}{\mu_j^2 K_2'(\alpha_j, s)(\beta_2\mu_j^4 + 1 - \delta_2)},$$

where

$$\Lambda(s) = \sum_{j=-2}^{\infty} \frac{(\alpha_j^2 + \nu s^2)^2}{K_2'(\alpha_j, s)(\beta_2\mu_j^4 + 1 - \delta_2)}, \quad \Upsilon(s) = \sum_{j=-2}^{\infty} \frac{\alpha_j^2[\alpha_j^2 + (2-\nu)s^2]^2}{K_2'(\alpha_j, s)(\beta_2\mu_j^4 + 1 - \delta_2)}.$$

It is shown (Marchenko, 1999) that the symmetric part of the diffraction-problem solution about scattering the flexural-gravity waves on the rectilinear crack has the waveguide mode. There is the value of $s = s_0 > \mu_0$ at which $\Lambda(s_0) = 0$. The value s_0 is extremely close to the value μ_0. Hence functions $a_0(s)$ and $a_2(s)$ have the pole in the absence of the rotational spring, that is at $k_2 = 0$. The residue at $s = s_0$ determines the amplitude of the waveguide mode, propagating along the crack and decreasing exponentially away from the crack.

Using the stationary phase method, we can find asymptotic behavior of the wave elevation amplitudes in the far field $|W(r, \theta)| = A(\theta)/\sqrt{r} + O(r^{-1})$ at $r \to \infty$ $(x = r\cos\theta,\ y = \sin\theta)$. It is shown that there are the predominant directions of wave propagation in the far field. These directions are at the angle to the crack for non-identical plates and for the configuration (i). For the identical plates there is waveguide mode along the crack. Fig. 1(a) shows the directional diagram $A(\theta)$ for different frequencies (in s^{-1}). The isolines of deflection amplitude $|W|$ are presented in Fig. 1(b) at $\omega = 1\ s^{-1}$. The following input date are used: $E = 6\ GPa$, $\rho_0 = 1025\ kg/m^3$, $\rho = 922.5\ kg/m^3$, $\nu = 0.3$, $h_1 = 1\ m$, $h_2 = 2\ m$, $a = 25\ m$, $x_0 = 50\ m$, $H = 100\ m$. The edges of the plates are free.

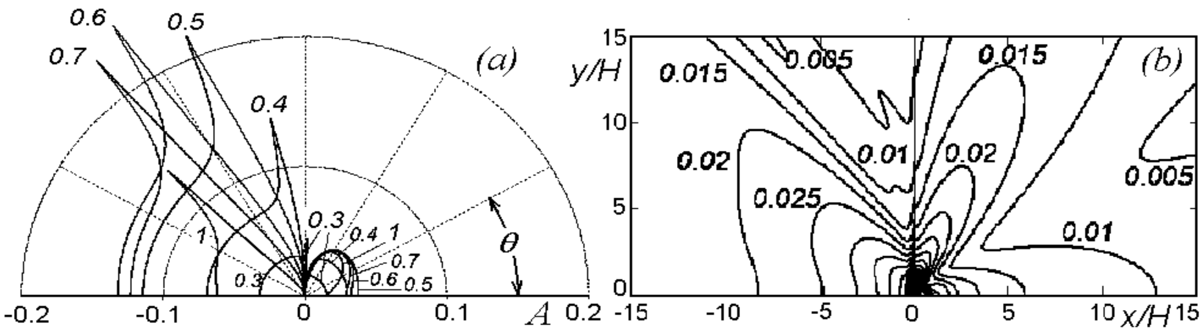

Figure. 1.

More detailed numerical results will be presented at the Workshop.

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