On the transport of energy by surface flexural-gravity waves

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1 INTRODUCTION

In order to build reliable models for sea state forecasting it is necessary to understand how the presence of floating sea ice influences the propagation of ocean waves, and vice versa. Here, and in the following text, the term "ocean waves" refers to either surface gravity waves in open water, or surface flexural-gravity waves in ice covered patches of the ocean. The interaction between waves and ice has to be understood on a local scale of individual ice floes, as well as on a global scale where, e.g., an entire marginal ice zone is simulated.

On the local scale the water body can be described as incompressible, inviscid fluid, while the ice cover is often approximated as a thin elastic plate or beam (depending on the dimension of the model), see, e.g. Kerr and Palmer (1972). If one assumes that wave steepness is sufficiently small, the equations governing the water surface can be linearised around the equilibrium position, which results in a linear scattering theory where the effect of each individual ice floe on wave propagation is completely described by its scattering matrix (see, e.g., Squire (2007)). Multiple scattering theory (see, e.g., Montiel et al. (2016)) is often used in an attempt to make predictions on larger scales on the basis of this local theory. This approach is computationally expensive, however, as large ensembles of randomly arranged ice floes have to be simulated.

On the global scale one usually considers an energy balance equation, here given in the one-dimensional form

$$(\partial_t + C_g\,\partial_x)I(t,x,k) = S_{\text{ice}} + S_{\text{nl}} + S_{\text{etc}}\,, \tag{1}$$

to describe the transport of energy through the field of ice floes, where $I(t,x,k)$ is the wave intensity that describes the directional rate of flow of energy density, k is the wavenumber, and C_g the group velocity of the wave (see, e.g., Meylan and Masson, 2006). The terms S_{ice}, S_{nl}, and S_{etc} describe the effect of the sea ice, non-linearity, and other processes, respectively, and their specific form depends on the particular model that is considered. The energy balance equation (1) is essentially the Boltzmann equation, as it is known e.g. from statistical mechanics.

To our knowledge, the precise relation between the local potential flow theory and the global energy balance models is not well understood, as the energy balance equation is usually derived in a hand-waving manner (see, e.g., Meylan and Masson (2006) and references therein). In this presentation I will discuss how this missing link may be established.

2 SURFACE FLEXURAL-GRAVITY WAVES IN VARYING ICE CONDITIONS

Consider an infinite ocean of constant depth H. When in equilibrium, it is bounded vertically by the sea floor at $z = -H$ and an ice cover at $z = 0$. The ice is modelled as a thin elastic beam (we work in 1+1 dimensions) with constant rheological parameters. A train of flexural-gravity waves is travelling at the water–ice interface in, say, the x-direction. Eventually, we want to describe the evolution of the amplitude envelope of this wave train when the ice cover thickness $h(x)$ varies in the x-direction. This notwithstanding, in the present abstract we only consider the case where $\partial_x h = 0$, since the x-dependent case is work in progress.

2.1 Linearised Potential Flow Equations

We begin with the description of water and sea ice on a local scale. As mentioned in the Introduction, we assume that the water body is an inviscid, incompressible, and irrotational fluid. Therefore, it can be fully described by a velocity potential $\phi(t,x,z)$, which is defined such that the local fluid velocity is its gradient. Conservation of mass implies that ϕ satisfies the Laplace equation

$$(\partial_x^2 + \partial_z^2)\phi = 0\,, \tag{2}$$

throughout the fluid domain. At the horizontal boundaries, ϕ satisfies the conditions

$$\partial_z\phi = 0\,, z = -H\,, \tag{3}$$

at the rigid sea floor, and

$$\partial_x^2(\beta\, h^3\, \partial_x^2(\partial_z\phi)) + g\,\partial_z\phi + \rho\, h\,\partial_t^2(\partial_z\phi) + \partial_t^2\phi = 0\,, z = 0\,, \tag{4}$$

at the water–ice interface. Here $\beta = G\,(1+\nu)/(6\,\rho_w)$, G is the shear modulus of the ice, ν is Poisson's ratio, $\rho = \rho_i/\rho_w$, and ρ_w and ρ_i are the densities of sea water and sea ice, respectively. The boundary condition (4) describes the interaction of the water body with the elastic thin beam (see, e.g., Mosig et al. (2015) for details), and is only valid for waves of modest amplitude.

2.2 Multiple Scale Analysis

As a first step to bridge the local scale and global scale models we perform a multiple scale analysis similar to that of Mei et al. (2005), §2.4.1. We introduce a dimensionless scaling parameter $\varepsilon \ll 1$. The velocity potential ϕ now depends on the multi-scale variables $x_n = \varepsilon^n x$ and $t_n = \varepsilon^n t$, i.e.

$$\phi(t, x, z) \longrightarrow \phi(t_0, t_1, \ldots, x_0, x_1, \ldots, z)\ . \tag{5}$$

We consider waves which are locally time-harmonic with angular frequency ω and wavenumber k, but on larger spacial and temporal scales their amplitude envelope may vary, that is,

$$\phi(t_0, t_1, \ldots, x_0, x_1, \ldots, z) = \big(\psi_0 + \varepsilon\,\psi_1 + \varepsilon^2\,\psi_2 + \ldots\big)\, \mathrm{e}^{\mathrm{i}\,(k\,x - \omega\,t)}\,, \tag{6}$$

where

$$\psi_n = \psi_n(t_1, t_2, \ldots, x_1, x_2, \ldots, z), \qquad n = 0, 1, 2, \ldots\ . \tag{7}$$

Note that the ψ_n do not depend on x_0 and t_0. Substituting this into the Laplace equation (2) and collecting terms of different orders in ε gives

$$\mathcal{O}(\varepsilon^0) : \mathcal{L}\psi_0 = 0\,, \tag{8}$$
$$\mathcal{O}(\varepsilon^1) : \mathcal{L}\psi_1 = \mathcal{K}_1\psi_0\,, \tag{9}$$
$$\mathcal{O}(\varepsilon^2) : \mathcal{L}\psi_2 = \mathcal{K}_2\psi_0 + \mathcal{K}_1\psi_1\,, \tag{10}$$

where $\mathcal{L} = (-k^2 + \partial_z^2)$, $\mathcal{K}_1 = (-2\,\mathrm{i}\,k\,\partial_{x_1})$, and $\mathcal{K}_2 = (-2\,\mathrm{i}\,k\,\partial_{x_2} - \partial_{x_1}^2)$. We perform the same procedure with the linearised ice / water surface condition (4) and obtain

$$\mathcal{O}(\varepsilon^0) : \mathcal{B}\psi_0 = 0 \qquad (z = 0)\,, \tag{11}$$
$$\mathcal{O}(\varepsilon^1) : \mathcal{B}\psi_1 = \mathcal{C}_1\psi_0 \qquad (z = 0)\,, \tag{12}$$
$$\mathcal{O}(\varepsilon^2) : \mathcal{B}\psi_2 = \mathcal{C}_2\psi_0 + \mathcal{C}_1\psi_1 \qquad (z = 0)\,, \tag{13}$$

where $\mathcal{B} = (-\omega^2 + \gamma\,\partial_z)$, $\mathcal{C}_1 = (2\,\mathrm{i}\,\omega\,\partial_{t_1} + 2\,\mathrm{i}\,\nu/\omega\,\partial_{x_1}\partial_z + 2\,\mathrm{i}\,\rho\,\omega\,h\,\partial_{t_1}\partial_z)$, and $\mathcal{C}_2 = (2\,\mathrm{i}\,\omega\,\partial_{t_2} - \partial_{t_1}^2 + 2\,\mathrm{i}\,\nu/\omega\,\partial_{x_2}\partial_z + 3\,\nu/(\omega\,k)\,\partial_{x_1}^2\partial_z + 2\,\mathrm{i}\,\rho\,\omega\,h\,\partial_{t_2}\partial_z + 2\,\mathrm{i}\,\rho\,\omega\,h\,\partial_{t_1}\partial_z)$. Here we defined the ω and k-dependent quantities

$$\gamma = \beta\,k^4\,h^3 - \rho\,\omega^2\,h + g\,, \tag{14}$$
$$\nu = 2\,\beta\,h^3\,k^3\,\omega\ . \tag{15}$$

Finally, the multiple scale expansion of the sea floor condition (3) is $\partial_z\psi_n(-H) = 0$ for all orders n.

The general solution of the $\mathcal{O}(\varepsilon^0)$ equations is

$$\psi_0 = \frac{g}{\omega}\,\cosh(k\,(z + H))\,\mathrm{sech}(k\,H)\,A\,, \tag{16}$$

where we chose the normalizing factor to ensure that the unknown amplitude $A = A(t_1, t_2, \ldots, x_1, x_2, \ldots)$ has units of length. The local wavenumber k has to be a solution to the dispersion relation

$$\omega^2 = k\,\gamma\,\tanh(k\,H)\ . \tag{17}$$

To find the solution ψ_1 to the first order equations, we subtract ψ_1 times equation (8) from ψ_0 times (9) and integrate over z, which gives

$$\int_{-H}^{0} \big(\psi_0\,\mathcal{L}\psi_1 - \psi_1\,\mathcal{L}\psi_0\big)\,dz = \int_{-H}^{0} \psi_0\,\mathcal{K}_1\psi_0\,dz\,, \tag{18}$$

The Fredholm alternative theorem implies that the solution ψ_1 can only exist if (18) is satisfied. After applying Green's identity the latter becomes

$$\Big[\psi_0\,\partial_z\psi_1 - \psi_1\,\partial_z\psi_0\Big]_{-H}^{0} = \int_{-H}^{0} \psi_0\,\mathcal{K}_1\psi_0\,dz\,, \tag{19}$$

and we can use the horizontal boundary conditions (12) at $z = 0$ and that $\partial_z\psi_1 = 0$ at $z = -H$, and substitute the general solution (16) of ψ_0 to obtain the first order solvability condition

$$\partial_{t_1}A + C_g\,\partial_{x_1}A = 0\,, \tag{20}$$

where $C_g = d\omega/dk$ is the group velocity. Equation (20) describes a wave package that moves in x-direction at the group velocity C_g without changing its shape. If $h \to 0$, equation (20) reduces to the equation found by Mei et al. (2005).

A similar procedure yields the second order solvability condition

$$\partial_t A + C_g\;\partial_x A = \frac{\mathrm{i}}{2\,\mu}\,\partial_x^2 A\quad . \tag{21}$$

where $\mu^{-1} = d^2\omega/dk^2$. Equation (21) resembles the Schrödinger equation known from quantum mechanics, in a reference frame moving with a speed C_g. The quantity μ is usually interpreted as an effective inertial mass of a particle which has a wave function that evolves according to (21). In the limit of a vanishing ice cover ($h \to 0$) equation (21) is identical to the result derived by Mei et al. (2005).

3 THE ENERGY BALANCE EQUATION FOR CONSTANT ICE COVER

In the previous section we derived the evolution equation (21) of the wave amplitude $A(t, x)$. Now we want to convert this into an evolution equation for the wave energy density. This is typically done by introducing the Wigner transform of A, named after E. Wigner (1932), which is defined as

$$W(t, x, k) = (2\,\pi)^{-1}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\,k\,y}\,A\big(t, x - y/2\big)\,A^*\big(t, x + y/2\big)\,dy\,, \tag{22}$$

where A^* is the complex conjugate of A. Watson and West (1975) have used this technique to derive the energy balance relation in deep water without an ice cover, but with non-linear terms and taking into account wind and surface currents. Unfortunately, their deep water equations require that the surface elevation is a smooth function which will not be the case once we investigate an ice cover that is broken up into multiple floes. A more general derivation of transport equations for waves in random media has been published by Ryzhik et al. (1996). Note, that

$$\int_{-\infty}^{+\infty} W(t, x, k)\,dk = \big|A(t, x)\big|^2\quad . \tag{23}$$

The Wigner distribution $W(t, x, k)$ can therefore be interpreted as (being proportional to) the energy density of the waves, provided that it is positive definite.

Using (21) and its complex conjugate we can find the energy balance relation for a constant ice cover

$$(\partial_t + C_g\,\partial_x)W(t, x, k) + \frac{k}{\mu}\,\partial_x W(t, x, k) = 0\,, \tag{24}$$

where we had to assume that $A(t, x) \to 0$ as $x \to \pm\infty$. Equation (24) is exactly the Boltzmann equation without forcing or scattering terms. Its left hand side only differs in the term proportional to μ^{-1} from the left hand side of the energy balance equation (1) that is typically used.

4 CONCLUSIONS AND FUTURE WORK

Using a multiple scale expansion of the Laplace equation and the linearised interface condition we have derived a Schrödinger-like equation which describes the time evolution of the amplitude envelope of a flexural-gravity wave train propagating in the x-direction. We have then used a Wigner distribution to derive the energy balance equation for this case. We emphasize the appearance of the effective mass μ in this relation, which does not appear in the energy balance equation that is commonly used by the ocean modelling / sea ice community.

Preliminary work suggests that when the ice cover thickness h is allowed to depend on x, equation (21) is amended by a potential term. This potential term would then appear as a scattering term in the energy balance equation. In this way, we hope to link ice cover inhomogeneities directly to a scattering term in the energy balance equation. This procedure is well known for other types of waves, as has been demonstrated by Ryzhik et al. (1996), but, to our knowledge, has never been performed for surface flexural-gravity waves.

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