Modelling Wave–Ice Shelf Interactions

M. H. Meylan[1], L. G. Bennetts[2], R. J. Hosking[2], and O. V. Sergienko[3]

[1]School of Mathematical and Physical Sciences, University of Newcastle, Australia
[2]School of Mathematical Sciences, University of Adelaide, Adelaide, SA 5005, Australia.
[3]Princeton University/Geophysical Fluid Dynamics Laboratory, Princeton, NJ, USA
e-mail address: mike.meylan@newcastle.edu.au

HIGHLIGHTS

A model for wave interaction with an ice shelf is presented and the theory is developed using Lax-Phillips scattering. We present results which show that the solution can be found in terms of modes of vibration which exist as zeros of the scattering matrix.

1 INTRODUCTION

Ice shelves are floating glaciers which form in the Arctic and Antarctic. Recent measurements have shown that waves generated by storms at distant continental coasts impact Antarctic ice shelves (Bromirski *et al.*, 2015). Modelling the wave–ice shelf system is hence of considerable importance. Many of the methods developed in offshore engineering can be applied, especially those developed in hydroelasticity. Here, we present preliminary results, using a model based on combined shallow-water and thin-plate theories (e.g. Holdsworth & Glynn, 1978; Sergienko, 2013).

2 Mathematical Model

A shelf of length L and uniform thickness $h \ll L$ floats on a water cavity of uniform depth H. The coordinate x denotes horizontal locations along the shelf/cavity, with its origin set to coincide with the seaward end of the shelf and $x = -L$ denoting the landward end. Open water of depth H exists for $x > 0$. As the wavelengths are assumed to be far greater than the water depth and the wave steepness to be small, the potential satisfies the linear shallow-water equation

$$\partial_x^2 \Phi = \frac{-1}{H} \partial_t \eta, \tag{1}$$

where $\eta(x, t)$ is the elevation of the water surface, and t denotes time. The function $\Phi(x, t)$ is the velocity potential, which satisfies a no-flux condition is applied at the landward end of the cavity

$$\partial_x \Phi = 0 \qquad \text{at} \qquad x = -L. \tag{2}$$

The ice-shelf is modelled as a thin–elastic plate, meaning its strain field can be determined from the displacement function satisfying

$$\begin{aligned} D\partial_x^4 \eta + \rho_{\mathrm{i}} h \partial_t^2 \eta + \rho_{\mathrm{w}} g \eta &= -\rho_{\mathrm{w}} \partial_t \Phi, \quad -L < x < 0, & (3) \\ \rho_{\mathrm{w}} g \eta &= -\rho_{\mathrm{w}} \partial_t \Phi, \quad x > 0, & (4) \end{aligned}$$

where the latter is the standard free-surface condition. Here $g \approx 9.81\,\mathrm{m\,s^{-2}}$ is the constant of gravitational acceleration, $\rho_{\mathrm{w}} \approx 1024\,\mathrm{kg\,m^{-3}}$ and ρ_{i} are water and ice densities, respectively, and

$D = Eh^3/\{12(1-\nu^2)\}$ is the the flexural rigidity of the shelf, where $E = 11\,\mathrm{GPa}$ is its effective Young's modulus and $\nu \approx 0.33$ its Poisson's ratio. The right-hand side denotes pressure forcing due to water motion in the cavity. The shelf is clamped at its landward end via the conditions

$$\eta = 0 \quad \text{and} \quad \partial_x \eta = 0 \quad \text{at} \quad x = -L, \tag{5a}$$

and free at its seaward end, with conditions

$$\partial_x^2 \eta = 0 \quad \text{and} \quad \partial_x^3 \eta = 0 \quad \text{at} \quad x = 0. \tag{5b}$$

At $x = 0$ we have the equations of continuity, applying the shallow draft approximation

$$\Phi(0^-, t) = \Phi(0^+, t), \quad \text{and} \quad \partial_x \Phi(0^-, t) = \partial_x \Phi(0^+, t). \tag{6}$$

A non-dimensionalisation is applied by defining

$$\hat{x} = \frac{x}{L_\mathrm{c}} \quad \text{and} \quad \hat{t} = \frac{t}{t_\mathrm{c}} \qquad \text{where} \qquad L_\mathrm{c} = \sqrt[4]{\frac{D}{\rho_\mathrm{w} g}} \quad \text{and} \quad t_\mathrm{c} = \sqrt{\frac{\rho_\mathrm{w} L_\mathrm{c}^6}{DH}}, \tag{7}$$

are the characteristic length and time, respectively. We write the non-dimensional equations as the abstract wave equation (following Hazard & Meylan, 2007)

$$\partial_t^2 \eta + \mathcal{A}\eta = 0, \tag{8}$$

where the operator $\mathcal{A}$ is given by

$$\mathcal{A}\eta = -\partial_x^2 \Psi. \tag{9}$$

Here Ψ is the negative acceleration potential $\Psi = -\partial_t \Phi$, which satisfies

$$-\eta + \Psi = \begin{cases} \partial_x^4 \eta + M \partial_x^2 \Psi & (-L < x < 0), \\ 0 & (x > 0), \end{cases} \tag{10}$$

where $M = \rho_\mathrm{i} h H / \rho_\mathrm{w} L_\mathrm{c}^2$. The operator $\mathcal{A}$ is self-adjoint and positive in the Hilbert space given by

$$\langle \eta, \eta' \rangle_{\mathcal{H}} = \langle \eta, \eta' \rangle_{[0,\infty]} + \langle \partial_x^2 \eta, \partial_x^2 \eta' \rangle_{[-L,0]} \tag{11}$$

where

$$\langle \eta, \eta' \rangle_{[a,b]} = \int_a^b \eta \, (\eta')^\star \, \mathrm{d}x. \tag{12}$$

and the star denotes complex conjugate.

3 SCATTERING MATRIX AND LAX-PHILLIPS THEORY

We assume that all terms are proportional to $\exp(-\mathrm{i}\omega t)$ and that the solution for $x > 0$ has the form

$$\eta = \mathrm{e}^{-\mathrm{i}kx} + R(\omega)\mathrm{e}^{-\mathrm{i}kx}, \tag{13}$$

where $k = \omega$ in our non-dimensional case. The reflection coefficient, R, is the scattering matrix in this system. The solution in the ice-shelf/cavity interval is

$$\partial_x^6 \Psi + (1 - M\omega^2)\partial_x^2 \Psi + \omega^2 \Psi = 0 \tag{14}$$

which can be solved exactly giving six unknowns. The value of these and the coefficient R are found by the seven boundary conditions. This method goes back to Stoker (1957).

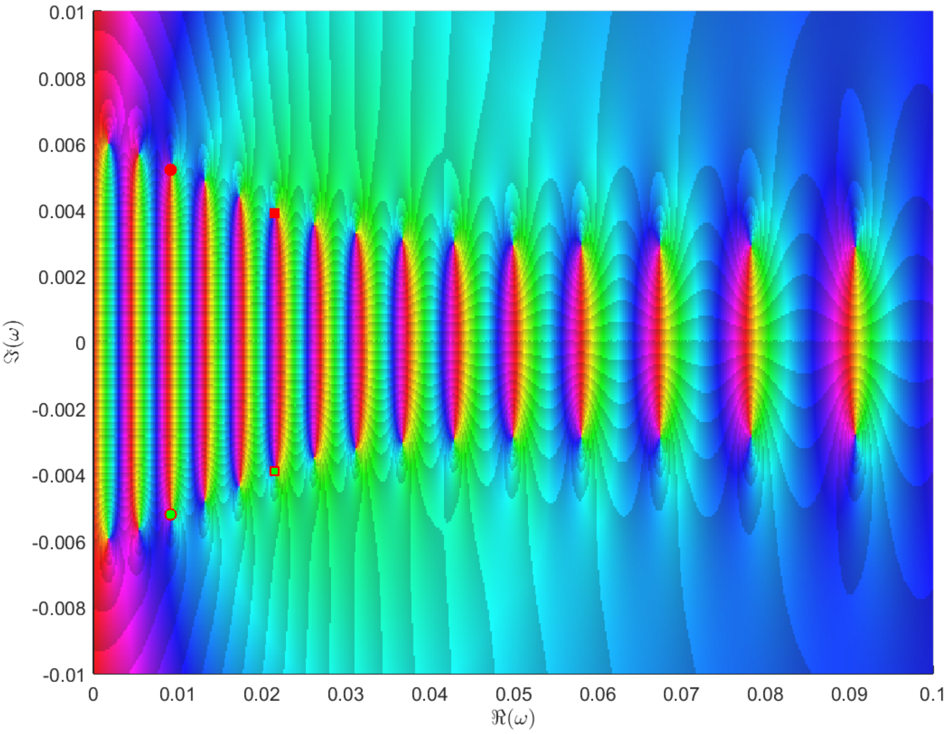

Figure 1: Visualisation of the scattering matrix $R(\omega)$ in the complex plane using the method of Wegert (2012). The colour represents the phase and the hue is proportional to the logarithm of the modulus. Rapid changes occur around the poles and zeros.

The problem can be described by the scattering theory of Lax & Phillips (1989) as was outlined in Meylan (2002). To apply Lax-Phillips scattering the following conditions are required: (i) the incoming and outgoing subspaces are orthogonal; and (ii) the incoming subspace spans the entire space under temporal evolution. These conditions are met by our system. Many consequences follow when Lax-Phillips scattering applies. The most significant is that the semi-group formed by the restriction of the temporal evolution operator to the subspace in the ice-shelf/cavity interval has point spectra with the same values as the singularities of the analytic extension of the scattering matrix $R(\omega)$. This in turn allows the solution to be given in terms of modes of vibration as for a self-adjoint operator.

4 RESULTS

We consider typical values for an ice-shelf and compute the analytic extension of the scattering matrix (or reflection coefficient). Figure 1 shows the analytic extension of the scattering matrix (or reflection coefficient) for $L = 40\,\text{km}$, $H = 200\,\text{m}$ and water depth $h = 300\,\text{m}$. The pattern of zeros and singularities at complex conjugates leads to the scattering matrix having the form of a Blaschke product. The green circle and square identify two of the singularities at $\omega = 0.0092 - 0.0052\text{i}$ and $\omega = 0.0216 - 0.0039\text{i}$, and the red circle and square identify the corresponding zeros at $\bar{\omega}$. Associated with each of these singularities/zeros are modes that form a biorthogonal system in which the solution can be expanded. The mode shapes associated with the singularity at $\omega = 0.0092 - 0.0052\text{i}$ and at $\omega = 0.0216 - 0.0039\text{i}$ are given in Figure 2. Further results will be presented at the workshop.

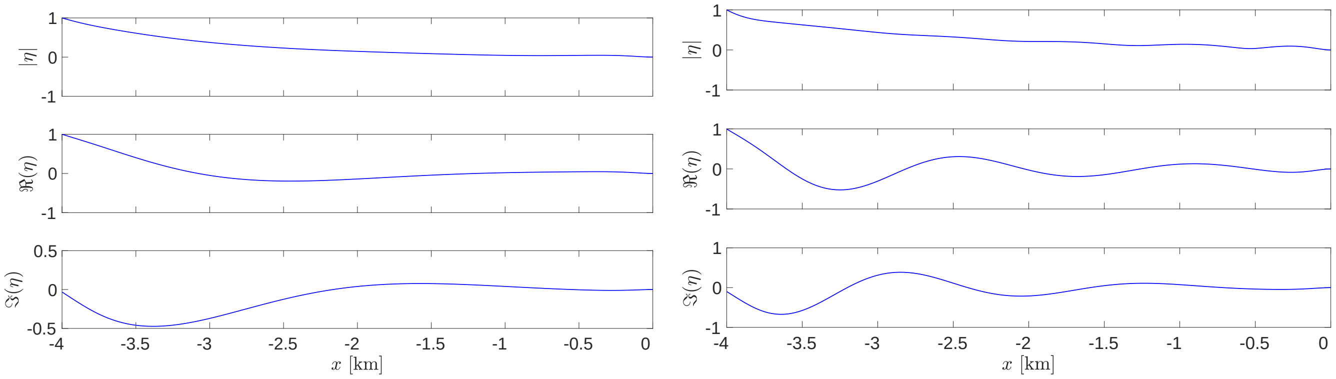

Figure 2: The mode shape associated with the singularity at $\omega = 0.0092 - 0.0052\mathrm{i}$, shown in Figure 1 by the green circle (left-hand plot) and mode shape associated with the singularity at $\omega = 0.0216 - 0.0039\mathrm{i}$, shown in Figure 1 by the green square (right-hand plot).

5 CONCLUSIONS

We have shown that a simple model for wave–ice shelf interactions leads to interesting results with important geophysical applications. We hope this work will motivate further study, especially the development of sophisticated hydroelastic models (e.g. Sergienko, 2010) including more realistic geometries.

ACKNOWLEDGEMENTS

This material is based upon research supported in part by the US Office of Naval Research under award number N00014-15-1-2611 (MHM).

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