Scattering of Flexural–Gravity Waves by a Group of Elastic Plates Floating on the Stratified Ocean with Multiple-layer Fluids

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1 INTRODUCTION

Within the frame of linear potential theory, a generalized multi-modular model composed of multiple elastic plates floating on the stratified ocean with multiple-layer fluids is derived. For the case of multiple plates, the well-developed methods in the literatures, e.g., Fox and Squire (1990, 1994) and Sahoo et al. (2001), will not be applicable. We will use a set of vertical eigenfunctions of free-surface waves to make inner products and obtain convergent numerical results. By the virtue of this method, the impact of ocean stratification on the inner forces of floating elastic plates is discussed via comparing a 4-layer fluid model with an 8-layer one.

2 MATHEMATICAL FORMULATION

We consider a generalized situation that N finite elastic plates with variable properties floating on a M-layer fluid, which can be seen as a multi-modular very large floating structure (VLFS) on the stratified ocean. The subscripts $n = 1, 2, \cdots, N$ and $m = 1, 2, \cdots, M$ are applied to mark each single plate and fluid layer, respectively. The elastic plates are continuously placed on the right side of the z-axis, as shown in Fig. 1. The length and midpoint of the n-th plate are assigned with $2L_n$ and $(c_n, 0)$. The flexural rigidity and the mass per unit length are denoted by D_n and M_n. The positions of every matching boundary from left to right are denoted by $x = a_0, a_1, a_2, \cdots, a_{N-1}, a_N$, where $a_0 = 0$. The density and thickness for the m-th layer are given by ρ_m and h_m, and then every interface as well as the seabed are located at $z = -H_m = -(h_1 + \cdots + h_m)$.

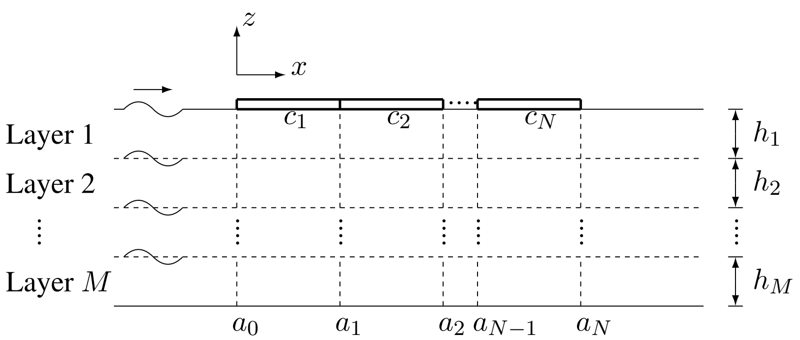

Fig. 1: Flexural–gravity wave scattering by multiple elastic plates floating on the stratified fluid with multiple layers

2.1 POTENTIAL FUNCTION

The linear potential theory is employed for this problem. With focusing on a specific frequency ω, the wave motion can be described by a velocity potential $\phi(x, z, t) = \Re[\Phi(x, z)\mathrm{e}^{-\mathrm{i}\omega t}]$, where $\Phi(x, z)$ is the spatial potential function. In the whole fluid domain, $\Phi(x, z)$ obeys the governing equation of $\nabla^2\Phi(x, z) = 0$.

The boundary conditions on the surface, every interface and bottom are formulated for $-\infty < x < +\infty$ as follows:

$$\rho_1\omega^2\Phi - \left(D_n\frac{\partial^4}{\partial x^4} - M_n\omega^2 + \rho_1 g\right)\frac{\partial\Phi}{\partial z} = 0, \qquad (z = 0), \tag{1}$$

$$\gamma_m\left[K\Phi - \frac{\partial\Phi}{\partial z}\right]_{z=-H_m^+} = \left[K\Phi - \frac{\partial\Phi}{\partial z}\right]_{z=-H_m^-}, \qquad (m = 1, 2, \cdots, M-1), \tag{2}$$

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=-H_m^+} = \left.\frac{\partial\Phi}{\partial z}\right|_{z=-H_m^-}, \qquad (m = 1, 2, \cdots, M-1), \tag{3}$$

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=-H_M} = 0, \tag{4}$$

where g is the gravitational acceleration, $\gamma_m = \rho_m/\rho_{m+1}$ and $K = \omega^2/g$. For the free surface areas, D_n and M_n vanish.

2.2 METHOD OF SOLUTION

By substituting the general solution of Laplace's equation into the boundary conditions, the vertical eigenfunction $V(k, z)$ can be deduced, which is a piecewise one with respect to every fluid layer:

$$V(k, z) = A_m \cosh k(z + H_m) + B_m \sinh k(z + H_m), \qquad (-H_m < z < -H_{m-1}), \tag{5}$$

where A_m and B_m are the coefficients relaying on the matrix iterative equations as follows:

$$\begin{pmatrix} A_m \\ B_m \end{pmatrix} = \cosh kh_{m+1} \begin{pmatrix} \dfrac{1}{\gamma_m} - \dfrac{\varepsilon_m k t_{m+1}}{\gamma_m K} & \dfrac{t_{m+1}}{\gamma_m} - \dfrac{\varepsilon_m k}{\gamma_m K} \\ t_{m+1} & 1 \end{pmatrix} \begin{pmatrix} A_{m+1} \\ B_{m+1} \end{pmatrix}, \tag{6}$$

with $A_M = \cosh^{-1} kH_M$, $B_M = 0$; k for wave numbers, $m = 1, \cdots, M-1$, $t_m = \tanh kh_m$, and $\varepsilon_m = 1 - \gamma_m$. The dispersion relation for every region is denoted by

$$\begin{pmatrix} t_1 C_n - K, C_n - K t_1 \end{pmatrix} \prod_m \begin{pmatrix} \dfrac{1}{\gamma_m} - \dfrac{\varepsilon_m k t_{m+1}}{\gamma_m K} & \dfrac{t_{m+1}}{\gamma_m} - \dfrac{\varepsilon_m k}{\gamma_m K} \\ t_{m+1} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \tag{7}$$

where $C_n = kF_n - KG_n$, with $F_n = D_n k^4/\rho_1 g + 1$, $G_n = M_n k/\rho_1$ for each elastic plate region and $F_n = 1$, $G_n = 0$ for the free surface regions; $\prod_m$ is the multiplication sign of $m = 1, 2, \cdots, M-1$ for the matrixes. For a given ω, the wave numbers can be sought out empirically. For the elastic plate region, we can find $2M$ real roots $\pm\tilde{k}_{n,0_m}$ $(m = 1, 2, \cdots, M)$, two couples of complex conjugates $\pm\mathrm{i}\tilde{k}_{n,j}$ $(j = \mathrm{I}, \mathrm{II})$ and infinite numbers of pure imaginary roots $\pm\mathrm{i}\tilde{k}_{n,j}$ $(j = 1, 2, \cdots)$ from Eq. (7), while for the free surface regions, we can find $2M$ real roots $\pm k_{0_m}$ $(m = 1, 2, \cdots, M)$ and infinite numbers of pure imaginary roots $\pm\mathrm{i}k_i$ $(i = 1, 2, \cdots)$.

The potential function should be separated for every region as

$$\Phi(x, z) = \begin{cases} \Phi_{I_0}(x, z) + \Phi_{R_0}(x, z), & (x < 0), \\ \Phi_{T_n}(x, z) + \Phi_{R_n}(x, z), & (a_{n-1} < x < a_n, n = 1, \cdots, N), \\ \Phi_{T_{N+1}}(x, z), & (x > a_N), \end{cases} \tag{8}$$

where the components, according to relevant wave numbers, are expressed via

$$\Phi_{I_0}(x, z) = \sum_m I_{0,0_m} \mathrm{e}^{\mathrm{i}k_{0_m} x} Z_{0_m}, \tag{9}$$

$$\Phi_{R_0}(x, z) = \sum_m R_{0,0_m} \mathrm{e}^{-\mathrm{i}k_{0_m} x} Z_{0_m} + \sum_i R_{0,i} \mathrm{e}^{k_i x} Z_i, \tag{10}$$

$$\Phi_{T_n}(x, z) = \sum_m T_{n,0_m} \mathrm{e}^{\mathrm{i}\tilde{k}_{n,0_m}(x-c_n)} \tilde{Z}_{n,0_m} + \sum_j T_{n,j} \mathrm{e}^{-\tilde{k}_{n,j}(x-c_n)} \tilde{Z}_{n,j}, \tag{11}$$

$$\Phi_{R_n}(x, z) = \sum_m R_{n,0_m} \mathrm{e}^{-\mathrm{i}\tilde{k}_{n,0_m}(x-c_n)} \tilde{Z}_{n,0_m} + \sum_j R_{n,j} \mathrm{e}^{\tilde{k}_{n,j}(x-c_n)} \tilde{Z}_{n,j}, \tag{12}$$

$$\Phi_{T_{N+1}}(x, z) = \sum_m T_{N+1,0_m} \mathrm{e}^{\mathrm{i}k_{0_m}(x-a_N)} Z_{0_m} + \sum_i T_{N+1,i} \mathrm{e}^{-k_i(x-a_N)} Z_i, \tag{13}$$

$$\left\{ Z_{0_m}(z), Z_i(z), \tilde{Z}_{n,0_m}(z), \tilde{Z}_{n,j}(z) \right\} = \left\{ V(k_{0_m}, z), V(\mathrm{i}k_i, z), V(\tilde{k}_{n,0_m}, z), V(\mathrm{i}\tilde{k}_{n,j}, z) \right\}, \tag{14}$$

$$I_{0,0_m} = -\mathrm{i}\omega\xi_m \left\{ \frac{\partial V(k_{0_m}, -H_{m-1})}{\partial z} \right\}^{-1}, \tag{15}$$

with ξ_m for the amplitudes of incident waves; $i = 1, 2, \cdots$, $j = \mathrm{I}, \mathrm{II}, 1, 2, \cdots$. Along every boundary between different regions ($x = a_0, a_1, \cdots, a_n, \cdots, a_N$), the potential expansions must satisfy the matching conditions as follow:

$$\Phi(a_n^-, z) = \Phi(a_n^+, z), \qquad \frac{\partial \Phi(a_n^-, z)}{\partial x} = \frac{\partial \Phi(a_n^+, z)}{\partial x}, \qquad (-H_M < z < 0). \tag{16}$$

The inner product method is employed for the M-layer fluid case and is defined as follows:

$$\langle U(z), V(z) \rangle = \sum_{m=1}^{M} \frac{\rho_m}{\rho_M} \int_{-H_m}^{-H_{m-1}} U \cdot V \, \mathrm{d}z, \tag{17}$$

where $U(z)$ and $V(z)$ represent arbitrary vertical eigenfunctions; for $m = 1$, $H_0 = 0$. Considering that the free-surface waves can be regarded as a limiting case from the flexural–gravity waves, we attempt to employ the vertical eigenfunctions of free-surface waves to make inner products for Eqs. (16). For this multiple layer case, an orthogonal relation between $Z_p(z)$ and $\tilde{Z}_{n,q}(z)$ is derived, which can help to transfer the inner product to an explicit differential expression:

$$\left\langle \tilde{Z}_{n,q}(z), Z_p(z) \right\rangle - \mathscr{D}_n(p,q) = 0, \quad \mathscr{D}_n(p,q) = \frac{(D_n \tilde{k}_{n,q}^4 - M_n \omega^2)}{\rho_M \omega^2 (k_p^4 - \tilde{k}_{n,q}^4)} \left[\frac{\partial \tilde{Z}_{n,q}}{\partial z} \frac{\partial^3 Z_p}{\partial z^3} + \frac{\partial^3 \tilde{Z}_{n,q}}{\partial z^3} \frac{\partial Z_p}{\partial z} \right]_{z=0},$$
$$\left(\begin{array}{l} p = 0_1, 0_2, \cdots, 0_M, 1, 2, \cdots \\ q = 0_1, 0_2, \cdots, 0_M, \mathrm{I}, \mathrm{II}, 1, 2, \cdots \end{array} \right). \tag{18}$$

Truncating the potential expansions in Eqs. (9)–(13) at $i = j = S$ and taking the vertical eigenfunctions of free-surface waves $Z_p(z)$ ($p = 0_1, 0_2, \cdots, 0_M, 1, 2, \cdots, S$) to make an inner product for each matching boundary, a matrix equation for the unknown coefficients is obtained as

$$\begin{pmatrix} \boldsymbol{M}_{R_0}^- & \boldsymbol{M}_{T_1}^+ & \boldsymbol{M}_{R_1}^+ & 0 & 0 & & 0 & 0 & 0 \\ \boldsymbol{N}_{R_0}^- & \boldsymbol{N}_{T_1}^+ & \boldsymbol{N}_{R_1}^+ & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \boldsymbol{M}_{T_1}^- & \boldsymbol{M}_{R_1}^- & \boldsymbol{M}_{T_2}^+ & \boldsymbol{M}_{R_2}^+ & & 0 & 0 & 0 \\ 0 & \boldsymbol{N}_{T_1}^- & \boldsymbol{N}_{R_1}^- & \boldsymbol{N}_{T_2}^+ & \boldsymbol{N}_{R_2}^+ & & 0 & 0 & 0 \\ & & \vdots & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & & \boldsymbol{M}_{T_N}^- & \boldsymbol{M}_{R_N}^- & \boldsymbol{M}_{T_{N+1}}^+ \\ 0 & 0 & 0 & 0 & 0 & & \boldsymbol{N}_{T_N}^- & \boldsymbol{N}_{R_N}^- & \boldsymbol{N}_{T_{N+1}}^+ \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}_{R_0} \\ \boldsymbol{\alpha}_{T_1} \\ \boldsymbol{\alpha}_{R_1} \\ \boldsymbol{\alpha}_{T_2} \\ \boldsymbol{\alpha}_{R_2} \\ \vdots \\ \boldsymbol{\alpha}_{T_N} \\ \boldsymbol{\alpha}_{R_N} \\ \boldsymbol{\alpha}_{T_{N+1}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\beta}_x \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \tag{19}$$

$\boldsymbol{M}_{R_0}^-$, $\boldsymbol{N}_{R_0}^-$, $\boldsymbol{M}_{T_{N+1}}^+$, and $\boldsymbol{N}_{T_{N+1}}^+$ are $(M+S)$ diagonal matrixes. According to Xu and Lu (2010), the inner products in these matrixes will be orthogonal for different wave numbers. For the same wave number, the diagonal elements are

$$\boldsymbol{M}_{R_0}^-(p) = -\boldsymbol{M}_{T_{N+1}}^+(p) = \mathscr{P}(p), \qquad \mathscr{P}(p) = \langle Z_p, Z_p \rangle, \tag{20}$$

$$\boldsymbol{N}_{R_0}^-(p) = \boldsymbol{N}_{T_{N+1}}^+(p) = \theta k_p \mathscr{P}(p), \qquad \theta = \begin{cases} 1, & (p = 0_1, 0_2, \cdots, 0_M), \\ \mathrm{i}, & (p = 1, 2, \cdots, S). \end{cases} \tag{21}$$

$\boldsymbol{M}_{T_n}^\pm$, $\boldsymbol{M}_{R_n}^\pm$, $\boldsymbol{N}_{T_n}^\pm$, and $\boldsymbol{N}_{R_n}^\pm$ are $(M+S)$ by $(M+S+2)$ matrixes:

$$\boldsymbol{M}_{T_n}^\pm(p,q) = \mp \mathrm{e}^{\mp \mathrm{i}\delta \tilde{k}_{n,q} L_n} \mathscr{D}_n(p,q), \qquad \boldsymbol{M}_{R_n}^\pm(p,q) = \mp \mathrm{e}^{\pm \mathrm{i}\delta \tilde{k}_{n,q} L_n} \mathscr{D}_n(p,q), \tag{22}$$

$$\boldsymbol{N}_{T_n}^\pm(p,q) = \pm \delta \tilde{k}_{n,q} \mathrm{e}^{\mp \mathrm{i}\delta \tilde{k}_{n,q} L_n} \mathscr{D}_n(p,q), \qquad \boldsymbol{N}_{R_n}^\pm(p,q) = \mp \delta \tilde{k}_{n,q} \mathrm{e}^{\pm \mathrm{i}\delta \tilde{k}_{n,q} L_n} \mathscr{D}_n(p,q), \tag{23}$$

$$\delta = \begin{cases} 1, & (q = 0_1, 0_2, \cdots, 0_M), \\ \mathrm{i}, & (q = \mathrm{I}, \mathrm{II}, 1, 2, \cdots, S). \end{cases} \tag{24}$$

$\boldsymbol{\alpha}_{R_0}$, $\boldsymbol{\alpha}_{T_{N+1}}$, $\boldsymbol{\beta}$, and $\boldsymbol{\beta}_x$ are $(M+S)$ dimensional column vectors; $\boldsymbol{\alpha}_{T_n}$ and $\boldsymbol{\alpha}_{R_n}$ are $(M+S+2)$ dimensional column vectors; the elements in these vectors are $\boldsymbol{\alpha}_{R_0} = [R_{0,0_1}, \cdots, R_{0,0_M}, R_{0,1}, \cdots, R_{0,S}]^\mathsf{T}$, $\boldsymbol{\alpha}_{T_n} = [T_{n,0_1}, \cdots, T_{n,0_M}, T_{n,\mathrm{I}}, T_{n,\mathrm{II}}, T_{n,1}, \cdots, T_{n,S}]^\mathsf{T}$, $\boldsymbol{\alpha}_{R_n} = [R_{n,0_1}, \cdots, R_{n,0_M}, R_{n,\mathrm{I}}, R_{n,\mathrm{II}}, R_{n,1}, \cdots, R_{n,S}]^\mathsf{T}$, $\boldsymbol{\alpha}_{T_{N+1}} = [T_{N+1,0_1}, \cdots, T_{N+1,0_M}, T_{N+1,1}, \cdots, T_{N+1,S}]^\mathsf{T}$, $\boldsymbol{\beta} = [-I_{0,0_1}\mathscr{P}(0_1), \cdots, -I_{0,0_M}\mathscr{P}(0_M), 0, 0, \cdots]^\mathsf{T}$, $\boldsymbol{\beta}_x = [k_{0_1} I_{0,0_1}\mathscr{P}(0_1), \cdots, k_{0_M} I_{0,0_M}\mathscr{P}(0_M), 0, 0, \cdots]^\mathsf{T}$.

Additional connection conditions are yet requested to complete a closed equation system for the calculation. For the generalized situation, four ideal connection conditions can be found at every joint between adjacent plates, and two ideal connection conditions can be found at each free edge. As a typical illustration, we use rotational springs with torsional rigidity J_n as connecting type to elaborate the process, then, associating with Eq. (19), $2(M+S)(N+1)+4N$ simultaneous equations for $2(M+S)(N+1)+4N$ unknown coefficients are established and the generalized problem can be solved afterwards.

3 DISCUSSIONS

Let ρ_1, H_M and $\sqrt{H_M/g}$ be respectively the characteristic quantities of density, length and time to transfer the problem to a nondimensionalized system. In order to study the impact of stratification on the mechanical behavior of a group of floating elastic plates, we try to use a parabolic curve as the approximation for the variation of density versus depth in the upper fluid areas and use a constant to denote a uniform layer below.

The density ρ_m versus the depth H_{m-1} $(m = 1, \cdots, M)$ follows the parabolic function of

$$\rho_m = -4.76\sigma H_{m-1}^2 + (\sigma + 0.2)\, H_{m-1} + 1, \qquad (\sigma = -0.2, 0, 0.2)\,, \tag{25}$$

where a parameter σ is used to simulate the profile of the curve. After taking several discrete points on the curve, we employ a 4-layer and an 8-layer fluid for the calculation. In the process, the 4-layer fluid is separated by the constants $h_1 = h_2 = h_3 = 0.07$ and $h_4 = 0.79$, while the 8-layer fluid is by $h_1 = \cdots = h_7 = 0.03$ and $h_8 = 0.79$. Other parameter configurations are $N = 4$, $D_1 = \cdots = D_4 = 0.05$, $M_1 = \cdots = M_4 = 0.0001$, $L_1 = \cdots = L_4 = 2$, $J_1 = J_2 = J_3 = 0.05$, and $\omega = 0.2$. In order to avoid the interference of internal waves, we assume the incident waves propagate only in a surface traveling mode, namely let $\xi_1 = 0.01$ and the other $\xi_n = 0$.

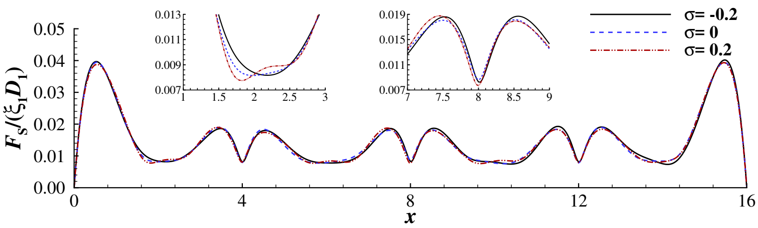

Fig. 2: Amplitude of shear force affected by different density distributions in a 4-layer fluid.

Figure 2 shows the calculation results for the amplitudes of shear force in the 4-layer case. A conspicuous variation is exhibited for the values along the whole structure, especially at the middle area of every single plate and the neighborhoods nearby the connections, which has been plotted in the subgraphs. The changing tends to be more intense in the 8-layer fluid. The ocean stratification will generate significant impact on the shear forces inside the elastic plates, especially in the position nearby the rotational springs.

4 CONCLUSIONS

We investigate a generalized VLFS–wave interaction model with multiple elastic plates floating on the stratified fluid of multiple layers, where the numbers of the plates as well as the layers can be arbitrary. Within the frame of linear potential theory, the inner product technique is used to deal with the matching relations. We introduce the vertical eigenfunctions of free-surface waves to make an inner product. Under this definition, an orthogonal relation with an explicit differential term is proposed. Investigations on the impact of stratified fluids are performed. For a floating structure with specified total length, we try to employ a parabolic curve to approximate the pycnocline in the ocean, where numbers of discrete points are drawn out as representatives for the density of every layer. The numerical results show the shear force will be greatly affected by different stratifications, which is a hypostatic distinction from a uniform fluid.

This research was sponsored by the National Natural Science Foundation of China under Grant No. 11472166.

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