# Breather propagation in shallow water

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## 1 Introduction

Breathers or envelope solitons correspond to a family of unstable and modulated wave trains, solutions of the Non Linear Schrödinger equation (NLSE). These envelope solitons are commonly used in various fields of physics, like optics, plasma, superfluids, solids or surface waves. Historically the first breather describes in the literature for the surface waves is the Peregrine breather (1983)[1]. It corresponds to a perturbation of the envelope with an infinite period in time and space. A second family of breathers widely describes in the literature is the Akhmediev breathers (Akhmediev (1986)[2] and (1993)[3]). These solutions are periodic in space with an infinite period in time. Theoretically these solutions exist only for values of $kh$ larger than a threshold equal to 1.363, with $h$ the water depth and $k$ the wavenumber. For values lower than this threshold, solitons become stable and perturbations are not longer amplified. These solutions are extensively studied because they are good approximations of modulated surface waves observed in the ocean.
The aim of this work is to propagate an Akhmediev breather over a variable bathymetry with a 1/200 slope. By selecting appropriate wave periods, wave trains evolve first in water depth where $kh$>1.363, and then up to the shore for values where $kh$<1.363. In the first part, the wave train is unstable, and the perturbation increases up to the focusing point, selected in our case for values of $kh$ near the threshold. Therefore when the wave train reaches the focusing distance, the amplitude of some waves of the group are much larger than the waves of the initial group. These waves can be identified as rogue waves. The question is to know if the rogue waves propagating in the shallow water region still exist or disappear.

## 2 Mathematical model

The mathematical model used to generate the modulated wave train is given by the NLSE. This equation describes the space-time evolution of the envelope amplitude $A(x,t)$ of a weakly nonlinear wave train propagating in various media. In arbitrary depth this equation is given by:

$$i\left(\frac{\partial A(x,t)}{\partial t}+c_g\frac{\partial A(x,t)}{\partial x}\right)-\alpha\frac{\partial^2 A(x,t)}{\partial x^2}-\beta|A(x,t)|^2A(x,t)=0 \tag{1}$$

where $c_g$ is the group velocity, $\alpha$ the dispersion coefficient and $\beta$ the nonlinearity coefficient (see [4] for details). This equation can be rewritten in a non dimensional form given by:

$$\mathrm{i}\,\psi_\xi+\psi_{\tau\tau}+2|\psi|^2\psi=0$$

$$\text{with}\quad \xi=x-C_gt,\quad \tau=-\alpha t\quad\text{and}\quad \psi(\xi,\tau)=\sqrt{\frac{\beta}{2\alpha}}A(\xi,\tau)$$

In the coordinate system $(\xi,\tau)$ propagating at the group velocity, a first order solution of this equation is given by Akhmediev [3]:

$$\psi\left(\tau,\xi\right)=a_0\frac{\sqrt{2\mathfrak{a}}\cos\left(\Omega\tau\right)+(1-4\mathfrak{a})\cosh\left(2R\xi\right)+\mathrm{i}R\sinh\left(2R\xi\right)}{\sqrt{2\mathfrak{a}}\cos\left(\Omega\tau\right)-\cosh\left(2R\xi\right)}\exp\left(2\mathrm{i}\xi\right)$$

$$\text{with} \quad \Omega = 2\sqrt{1-2\mathfrak{a}}, \quad R = \sqrt{8\mathfrak{a}(1-2\mathfrak{a}^2)} \quad \text{and} \quad \mathfrak{a} < 0.5$$

The parameter $\mathfrak{a}$ is the breather parameter. The envelope period is function of this parameter and the initial amplitude of the wave train $a_0$.

# 3 Experimental setup

These experiments were conducted at the Tainan Hydraulics Laboratory (THL) of the National ChengKung University in Taiwan in the basin called "Mid-size Observation Flume". The length of the basin is 200m, the width is 2m and the water depth was fixed to $h_0$=0.975m. One end of the basin is equipped with a piston wavemaker. The bathymetry consists in a constant part of 31m, a 1/4 slope over 20m and a 1/200 slope beyond. With the initial water depth set to 0.975m, the shoreline is at $L_S$=146m from the wavemaker.
To measure the evolution of the free surface, 49 capacitive wave gauges were installed, distributed from 4 to 131m. A drawing of the experimental setup with the bathymetry and the wave gauges location is given in figure 1.

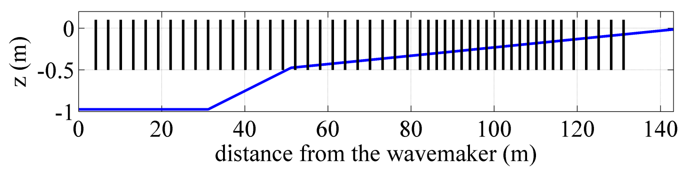


Figure 1: *Drawing of the experimental setup with the bathymetry and the wave gauges location.*

The experimental conditions correspond to different initial $k_0h_0$ and $a_0k_0$, with $k_0$ the wavenumber for the water depth $h_0$. These conditions correspond to 32 different values of the couple of parameters. For each values, one Akhmediev breather is generated with $\mathfrak{a}$=0.35 and a focusing distance equal to the water depth where $kh$=1.363. Also a corresponding regular wave train is generated with the same initial values in order to estimate the dissipation along the basin. This dissipation is due to friction on the walls and on the bottom and due to viscosity at the air-water interface. The dissipation is known as a key parameter regarding the stability of the wave train[5].

# 4 Evaluation of the dissipation

Following numerous authors (Lamb (1932)[6], Mei (1983)[7], Tulin & Waseda (1999)[8]), energy $E(x)$ of the wave train is supposed to decrease exponentially :

$$E(x) = E_0(x)\exp(-2\sigma\, x)$$

with $\sigma$ the dissipation rate. The figure 2 displays the dissipation rate as function of the wave period and the steepness. These results show that the dissipation increases when the wave period decreases but there is no direct relation with the steepness except for the smallest period.

These values of the dissipation will be used for the numerical simulations.

# 5 Results

## 5.1 Equation

To compare our results with a numerical code, a NLSE equation in variable bathymetry is considered. This equation was first developed by Djordjevic and Redekoop (1978)[9]. The following equation is the same as

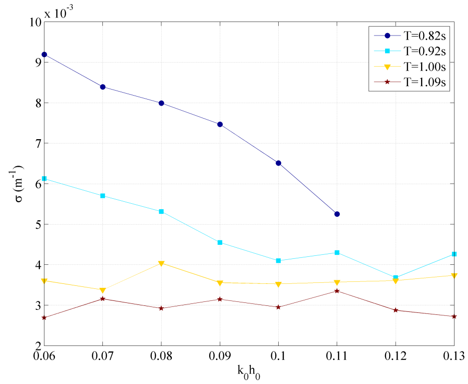


Figure 2: *Evolution of $\sigma$ as function of the wave period and the steepness.*

for the cited authors but with an adding term to take into account the linear dissipation:

$$\mathrm{i}\left(\frac{\partial A}{\partial x}+\frac{1}{C_g}\frac{\partial A}{\partial t}\right)=-\,\mathrm{i}\,\mu\frac{d(kh)}{dx}A+\lambda\frac{\partial^2 A}{\partial t^2}+\nu|A|^2A+\mathrm{i}\,\frac{\sigma}{C_g}\,A$$

$$\text{with}\quad \mu=\frac{(1-\sigma^2)(1-kh\,\sigma)}{\sigma+kh(1-\sigma^2)}\;,\quad \lambda=\frac{1}{2C_g\omega_0}\left[1-\frac{gh}{C_g^2}(1-kh\,\sigma)\left(1-\sigma^2\right)\right]\;,$$

$$\nu=\frac{\omega_0k^2}{16C_g\,\sigma^2}\left[9-10\sigma^2+9\sigma^4-\frac{2C_g^2\,\sigma^2}{gh-C_g^2}\left(4\frac{C_p^2}{C_g^2}+4\frac{C_p}{C_g}(1-\sigma^2)+4\frac{gh}{C_g^2}\left(1-\sigma^2\right)^2\right)\right]$$

and $\sigma$ the dissipation rate. This equation is solved with an usual split-step method.

### 5.2 A generic example

A typical example is presented in figure 3. This example corresponds to $k_0h_0 = 5.83$, $a_0k_0 = 0.10$, $\mathfrak{a}$=0.35 and a focusing distance for $(kh)$ equal to the threshold, here 76m. In this figure, the first column shows the experimental results. For the upper plot, the time evolution of the free surface is displayed with the corresponding envelope for 5 different wave gauges. Below the space-time representation of the envelope is showed. In these two plots, the x-axis is shifted according to the group celerity. The second column is the same representation but for the numerical code. First there is a very good agreement between experiment and simulation. Secondly, in both experiment and simulation, a second maximum of the envelope between the two initial ones is observed for a distance to the wavemaker larger than the distance corresponding to the threshold. This second maximum corresponds to an instability of the wave train that occurs on the shallow water region. This instability is due to dissipation.

The third column corresponds to the comparison between simulation and experiment for the space evolution of the maximum and the minimum of the positive part of the envelope. Again the agreement is excellent. The maxima increase up to the change of slope and then decrease. There is no noticeable modification when the train enter in the shallow water region. The minima decrease gradually up to the threshold and then become almost constant with a value near zero. It means that the wave train is composed by wave group with a restricted number of wave separated by time interval where there is almost no wave.

## 6 Conclusions

This work allowed to study the evolution of breathers in variable bathymetry. This is the first time this problem is studied by the scientific community. We showed that the breathers keep their peculiar structure

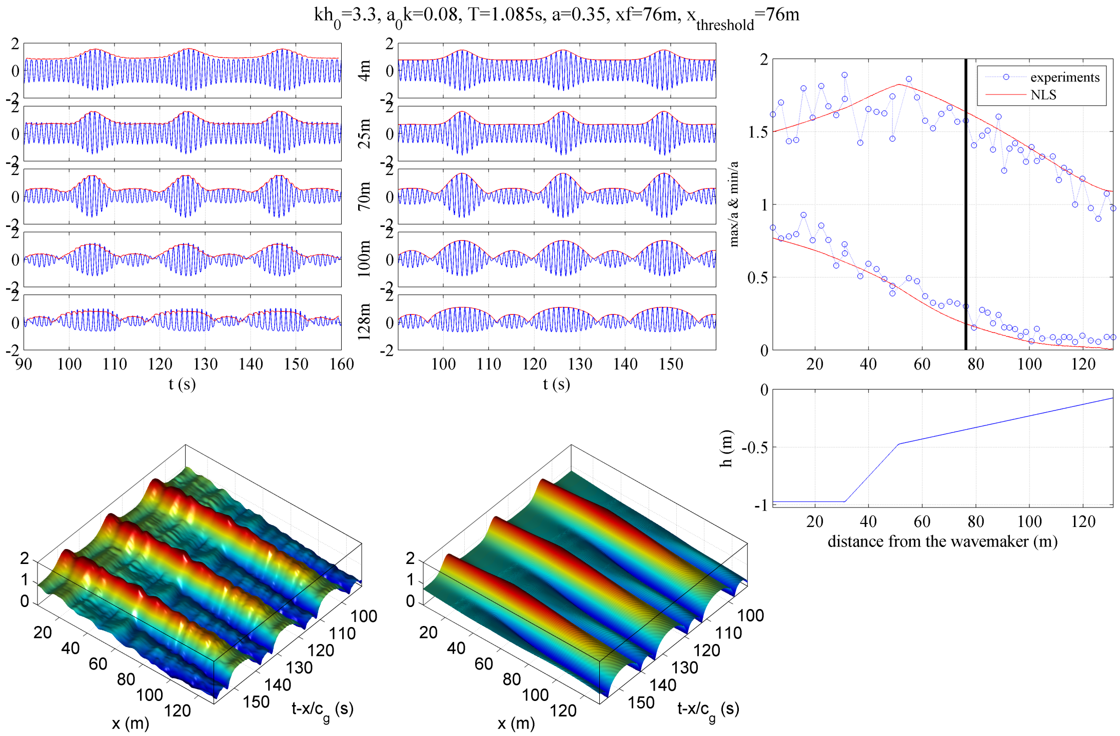


Figure 3: *The first column corresponds to the experimental data, the second to NLS equation in variable bathymetry and the third to the comparison between the maximum and the minimum of the upper envelope of the wave train.*

when they enter in the shallow water region. However the dissipation is important and part of the decrease of the wave train is due to this effect. But this dissipation leads also to instabilities conducting to less predictable behavior.

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