Design of Small Water Channel Network for Shallow Water Cloaking

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1 INTRODUCTION

Cloaking refers to a phenomenon that makes an object invisible. The research related to the cloaking has been undertaken, and current studies are influenced by Pendry *et al.* (2006). He proposed the theory based on the coordinate transformation and Schurig *et al.* (2006) demonstrated a feasible technique to realize the cloaking. In the theory, a space is deformed to mimic an object and make waves propagate to detour the cloaked space. As a result, a cloaked object does not scatter waves and behaves as if it is invisible (see Fig. 1 as an example of computed results). In fact, bending the physical space seems impossible. Alternatively, such a deformed space is interpreted as a physical space filling with anisotropic mediums which have properties of their equivalence. Since cloaking requires peculiar features which do not exist in the nature, such artificial materials are called meta-materials.

Recently, the concept of cloaking has been enthusiastically applied to other wave fields, such as acoustic wave (Cummer *et al.*, 2007 or Zigoneanu *et al.*, 2014) or seismic wave (Brule *et al.*, 2014). Similarly, some studies have been made in water waves; utilizing viscosity (Farhat *et al.*, 2008), water depth (Zareei *et al.*, 2015), elastic plate (Zareei *et al.*2016), and fluid density (Iida *et al.*, 2016). These methods are reviewed by Porter (2016).

Our study was intended to achieve the cloaking by controlling fluid density, however, we finally found it out to be inappropriate approach because of vertical restriction of water waves unlike electromagnetic or acoustic waves. Hence, we propose an alternative method, a small water channel network, to cloak an object from shallow water waves. Waves propagating to a water channel are analogous to electric currents flowing to an electric circuit and these properties are analytically designed by the approach of circuit theory (Mochizuki *et al.*, 1990). In an electric circuit, the anisotropy is represented by combination of different size of lines (Nagayama *et al.*, 2015). Especially, macro characteristics of electric waves are controllable if the circuit is sufficiently smaller than the wave length. It has an advantage of working for a wide range of frequency.

In this paper, we propose an analytical design of small water channel to induce anisotropic wave fields so as to accomplish the cloaking. By changing the widths and depth of small water channel, desirable properties are obtained. A design method of water channel for cloaking is indicated and computation results will be shown in the Workshop.

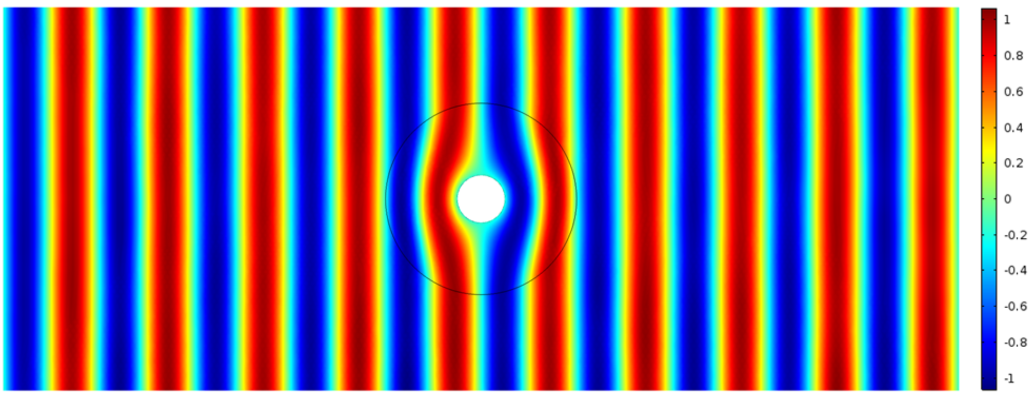


Fig. 1 Computation result of shallow water wave cloaking by special fluid.

2 THEORY

2.1 Equivalent circuit of waves in rectangular channel

Characteristics of waves propagating in a water channel can be expressed by equivalent electric circuits; the pressure and flow rate are related to a relation between voltage and electric current. Let us describe an equivalent circuit model of rectangular channel with the coordinate system shown in Fig. 2 (a). By assuming incompressible and inviscid flow with irrotational motion, a linearized potential flow is considered. Then the x-components of the continuity equation and Euler equations are expressed as

$$-\frac{\partial P}{\partial x} = \rho\frac{\partial u}{\partial t},\; -\frac{\partial u}{\partial x} = \frac{k_x^2}{\rho\omega^2}\frac{\partial P}{\partial t} \tag{1}$$

Here P is the pressure, ρ is the fluid density, ω is the circular frequency, and u and k_x are the x-components of fluid velocity and wavenumber, respectively. Under the long wave approximation (the water depth h is sufficiently smaller than the wave length λ), the wavenumber is obtained as

$$k_x = \sqrt{\frac{\omega^2}{gh} - \left(\frac{n\pi}{b}\right)^2} \tag{2}$$

where g is the gravity acceleration, and n is number of wave mode, and b is the channel width. Here we only treat the mode of $n = 0$, which is a wave propagating to the x-axis. By defining the flow rate as $Q = bhu$, x-components of Eq. 1 are rewritten as

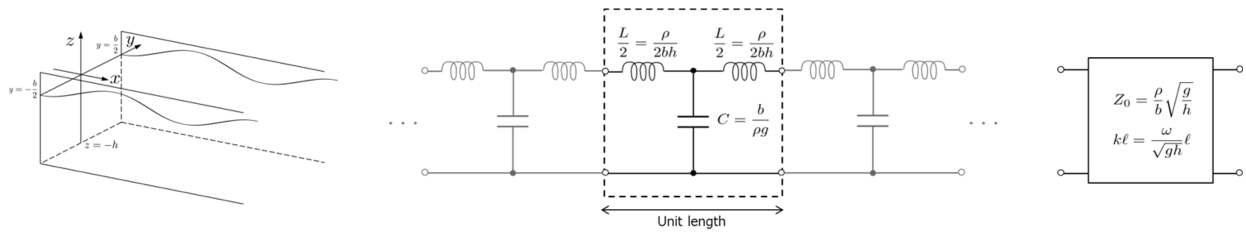
$$\frac{\partial P}{\partial x} = -i\omega\frac{\rho}{bh}Q,\; \frac{\partial Q}{\partial x} = -i\omega\frac{b}{\rho g}P \tag{3}$$

Eq. 3 is equivalent to the loss less circuit equations. Hence coefficients of Eq. 3, $\frac{\rho}{bh}$ and $\frac{b}{\rho g}$, correspond to inductance L and electric capacity C. Fig. 2 (b) shows an equivalent circuit model of channel per unit channel length. Characteristics of this channel are obtained by the circuit theory as follows:

$$\left.\begin{aligned} Z_0 &= \sqrt{\frac{i\omega L}{i\omega C}} = \frac{\rho}{b}\sqrt{\frac{g}{h}} \\ \gamma^2 &= i\omega L \times i\omega C = \frac{-\omega^2}{gh} \end{aligned}\right\} \tag{4}$$

where Z_0 is the impedance and γ is a propagation constant which is equal to ik. Actual water channel is of finite length and thus a lot of equivalent circuits are connected in cascade as Fig. 2 (b). Such a circuit with non-zero length are called the distributed constant circuit and represented as Fig. 2 (c). A relation between the pressure and flow rate at two points x and $x + 1$ with distance ℓ is analytically obtained as

$$\begin{pmatrix} P_x \\ P_{x+1} \end{pmatrix} = -iZ_0 \begin{pmatrix} \cot k\ell & \csc k\ell \\ \csc k\ell & \cot k\ell \end{pmatrix} \begin{pmatrix} Q_x \\ Q_{x+1} \end{pmatrix} \tag{5}$$



(a) (b) (c)

Fig. 2 Relations between rectangular shallow water channel and electric circuits. (a) view of rectangular channel (b) equivalent circuit model per unit length of channel (c) equivalent distributed constant circuit model for channel.

2.2 Small water channel network to cloak a cylinder

Since the Helmholtz equation is invariant from the coordinate transformation, transformation optics are adaptable for shallow water waves. Let us consider the linear coordinate transformation $r' = \frac{b-a}{b}r + a$ to compress a space from a cylindrical region to an annular region so as to cloak an object in the annulus, where a and b are inner and outer radii of the cloaking region, respectively. Here attention is focused on a micro section $\Delta r \times r\Delta\theta \times h_0$ that is selected by $\Delta r = r\Delta\theta$ in the polar coordinate system. If following governing equations are satisfied, the cloaking can be realized

$$\left.\begin{aligned} \frac{\partial P}{\partial r} &= -\frac{r}{r-a}\frac{\rho}{r\Delta\theta h_0}\frac{\partial Q_r}{\partial t} \\ \frac{1}{r}\frac{\partial P}{\partial \theta} &= -\frac{r-a}{r}\frac{\rho}{\Delta r h_0}\frac{\partial Q_\theta}{\partial t} \\ \frac{1}{r}\frac{\partial}{\partial r}(rQ_r) + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} &= -\left(\frac{b}{b-a}\right)^2\frac{r-a}{r}\frac{\Delta r}{\rho g}\frac{\partial P}{\partial t} \end{aligned}\right\} \tag{6}$$

In order to create wave fields governed by Eq. 6, we propose a small water channel network. The cloaking domain is discretized by natural numbers N_r for r direction and N_θ for θ direction, that is, totally $N_r \times N_\theta$ sections as Fig. 3 (a). Each section is composed of small water channel which satisfies Eq. 6 at the center of the channel. Small water channel is approximated with a rectangular channel with directions ξ ($\parallel r$) and η ($\parallel \theta$) in order to simplify the treatment. (Energy spreading along to the radial direction is ignored.) The width of channel in each direction are represented by b_ξ, b_η and the depth is h_1 as shown in Fig. 3 (b) and these are variables to be designed. Then, an equivalent distributed circuit model of small water channel as Fig. 3 (c) is obtained by Eqs. 3 and 5 and also by the circuit theory. As a result, following conditions are obtained:

$$\left.\begin{aligned} \frac{i\omega}{2}\left(\frac{r}{r-a}\frac{\rho}{h_0}\right) &= iZ_\xi \tan\frac{k_1\Delta r}{2} \approx \frac{i\omega}{2}\frac{\rho\Delta r}{b_\xi h_1} \\ \frac{i\omega}{2}\left(\frac{r-a}{r}\frac{\rho}{h_0}\right) &= iZ_\eta \tan\frac{k_1\Delta r}{2} \approx \frac{i\omega}{2}\frac{\rho\Delta r}{b_\eta h_1} \\ \frac{1}{i\omega}\left(\frac{b}{b-a}\right)^2\frac{r-a}{r}\frac{\rho g}{\Delta r^2} &= \frac{Z_\xi Z_\eta}{2i(Z_\xi+Z_\eta)\sin\frac{k_1\Delta r}{2}\cos\frac{k_1\Delta r}{2}} \approx \frac{\rho g}{i\omega\Delta r(b_\xi+b_\eta)} \end{aligned}\right\} \tag{7}$$

where the water channel is assumed to be smaller enough than the wave length and therefore it becomes independent of the frequency. Sorting Eq. 7, we get

$$\left.\begin{aligned} b_\xi &= \frac{b^2}{b^2-a^2}\frac{r^4}{(r^2-a^2)^2+r^4}\Delta r \\ b_\eta &= \frac{b^2}{b^2-a^2}\frac{(r^2-a^2)^2}{(r^2-a^2)^2+r^4}\Delta r \\ h_1 &= \frac{b^2-a^2}{b^2}\frac{(r^2-a^2)^2+r^4}{r^2(r^2-a^2)}h_0 \end{aligned}\right\} \tag{8}$$

By Eq. 7, each small water channel is designed according to its location. Small water channels are connected as a network and it works regardless of the frequency as long as the water channel can be treated as small.

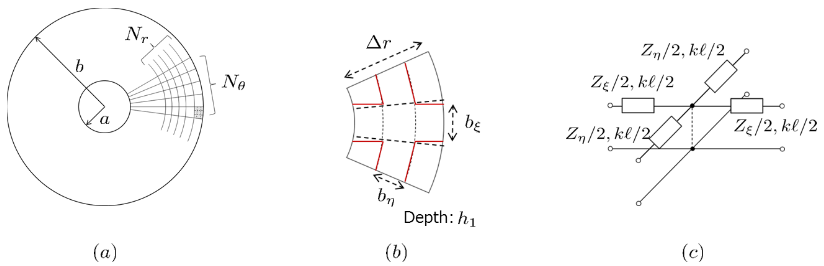


Fig. 3 Design of the cloaking space by small water channel network (a) discretization of cloaking domain (b) view of small water channel for cloaking (c) equivalent distributed constant circuit model.

Let us obtain the parameters of each water channel for shallow-water waves with depth h_0 for instance. The cloaking domain is modeled for outer radius $b = 21.7$ and inner radius $a = 5.0$. The number of partitions is set to $N_\theta = 60$ and then N_r is decided to be 14. Totally 840 water channels and variables (b_ξ, b_η, h_1) are designed by Eq. 7. Non-dimensional parameters are shown in Fig. 4.

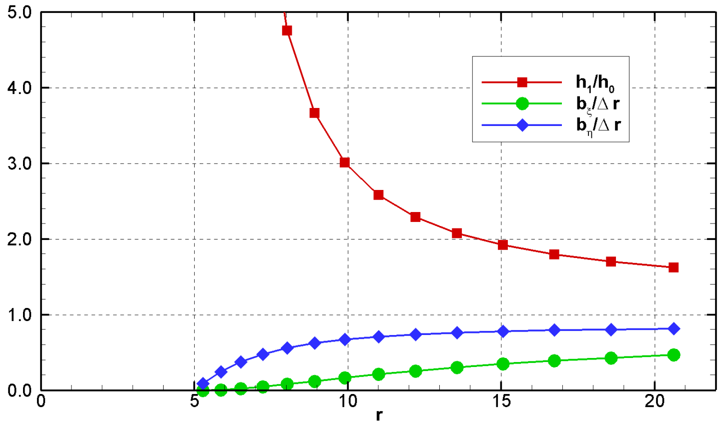


Fig. 4 Nondimensional parameters of each water channel along to the radial distance from the origin.

We can see that the values of width b_ξ and b_η are smaller than the channel size Δr through the cloaking domain. Since h_1 in Eq. 8 becomes infinity at $r = a$, the depth of channel increases along the negative radial direction. In addition, the water depth must be restricted in conformity to long-wave approximation. Therefore we must approximate the depths with practical values.

3 CONCLUSION

In this paper, we proposed a design method of small water channel network to achieve the cloaking in shallow water. First, an equivalent circuit model of waves in a rectangular channel was described. Equivalent properties of inductance and capacity were obtained by analogy between shallow-water waves and electric waves. Characteristics of water waves between arbitrary two points were also obtained. Second, the governing equations under cloaking and their equivalent water channels were calculated. The cloaking domain was discretized and the water channel was mounted in each section. Then, the widths in the orthogonal directions and the depth of each channel are analytically obtained. Since each water channel is very small, properties are independent of the frequency. Parameters of small water channels were shown as an example. Numerical computations by these water channels will be presented at the Workshop.

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