Extension of Haskind's relations to cylindrical wave fields in the context of an interaction theory

F. Fàbregas Flavià* and **A. H. Clément**

Ecole Centrale de Nantes, LHEEA Lab., UMR CNRS 6598, Nantes, France
*francesc.fabregas-flavia@ec-nantes.fr

1 INTRODUCTION

The finite-depth Direct Matrix Method interaction theory (IT) proposed by Kagemoto and Yue (1986) enables one to solve the multiple-scattering problem efficiently for large arrays of identical floating bodies. For that, two hydrodynamic operators known as Diffraction Transfer Matrix (DTM) and Radiation Characteristics (RC) which model the way a body scatters and radiates waves respectively were introduced.

The IT has been used in various contexts, such as hydrodynamic interactions between vessels (Chakrabarti, 2001), large fields of ice floes in the marginal ice zone (Peter and Meylan, 2004) and very large floating structures (Kashiwagi, 2000). McNatt et al. (2015) applied it to study large arrays of wave energy converters. They derived a novel method to compute the DTM and introduced an operator called Force Transfer Matrix (FTM) to facilitate the calculation of the forces exerted on the bodies.

In this paper, a novel set of relations linking the FTM to the RC components is obtained and numerically checked. They extend the classical Haskind's relations valid with incident plane waves to the cylindrical components of the scattered and radiated fields.

2 CYLINDRICAL WAVE FIELDS

One of the main pillars of the IT proposed by Kagemoto and Yue (1986) is that any diffracted or radiated wave field can be represented using a basis of cylindrical eigenfunctions. These functions, generally referred to as partial cylindrical waves, can be defined with respect to a local cylindrical reference system centered at the body. This series expansion is the sum of progressive waves, related to the far-field behaviour and evanescent standing waves, in the near-field. Only the former will be considered hereafter. In this base of cylindrical functions the different forms of the far-field potential, i.e. incident, scattered and radiated follow as:

$$\phi_i^I(r_i,\theta_i,z) = f(z)\,\Lambda^I(r_i,\theta_i);\quad \phi_i^{R,k}(r_i,\theta_i,z) = f(z)\,\Lambda^{R,k}(r_i,\theta_i);\quad \phi_i^S(r_i,\theta_i,z) = f(z)\,\Lambda^S(r_i,\theta_i) \tag{1}$$

$$f(z) = \frac{\cosh\, k_0(z+d)}{\cosh\, k_0 d};\quad \Lambda^I(r,\theta) = \sum_{m=-\infty}^{\infty} (a_i^I)_m\, J_m(k_0 r)\, e^{im\theta} \tag{2}$$

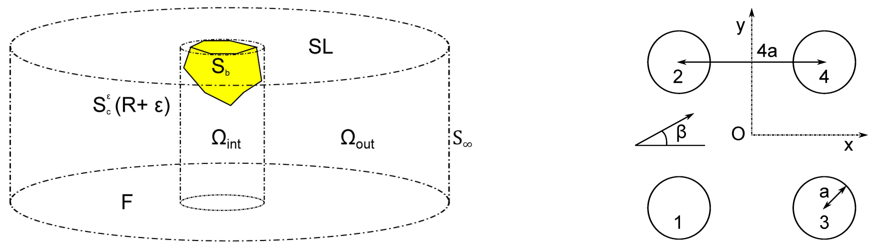
$$\Lambda^{R,k}(r,\theta) = \sum_{m=-\infty}^{\infty} (R_i^k)_m\, H_m(k_0 r)\, e^{im\theta};\quad \Lambda^S(r,\theta) = \sum_{m=-\infty}^{\infty} (A_i)_m\, H_m(k_0 r)\, e^{im\theta} \tag{3}$$

where ϕ_i^I is the incident potential to body i, $\phi_i^{R,k}$ the radiated potential due to a motion of body i in a degree of freedom k when all the other bodies are held fixed, ϕ_i^S the scattered potential, k_0 the progressive wave number, $H_m(k_0 r)$ the Hankel function of order m, $J_m(k_0 r)$ the Bessel function of the first kind of order m, d the depth and $(a_i^I)_m$, $(R_i^k)_m$ and $(A_i)_m$ the m^{th} components of the partial wave cylindrical coefficients vector of the incident, radiated and scattered potentials respectively.

2.1 Isolated body

The excitation force (F_{ex}^k) can be computed from incident and radiated potentials only using Haskind's relation (Newman, 1962):

$$F_{ex}^k = i\omega\rho \int_{S_b} \left(\phi^I \frac{\partial \phi^{R,k}}{\partial n} - \phi^{R,k} \frac{\partial \phi^I}{\partial n} \right) dS = \sum_{m=-\infty}^{\infty} \mathbf{G}_m^k a_m^I \tag{4}$$



(a) (b)

Figure 1: **(a) Schematic of the fluid domains. Free surface (SL); body's wetted surface (S_b); body's circumscribing cylinder radius (R_c); cylindrical surface infinitesimally larger than the body's circumscribing cylinder (S_c^ϵ), seabed (F); limit of the domain at infinity (S_∞); (b) Schematic of a 4 cylinder array.**

where $\mathbf{G}_m^k$ corresponds to the m^{th} element of the k^{th} row of the FTM and a_m^I represents the m^{th} term of the ambient incident wave cylindrical coefficients vector.

The expressions of the incident, scattered and radiated potentials using the base of partial cylindrical wave functions are valid only outside the circumscribing cylinder of the body (Ω_{out}). Thus, and by means of Green's theorem, we express the surface integral in expression (4) on a cylindrical control surface (S_c^ϵ) infinitesimally larger than the body's circumscribing cylinder (Figure 1):

$$\int_{S_b} \left(\phi^I \frac{\partial \phi^{R,k}}{\partial n} - \phi^{R,k} \frac{\partial \phi^I}{\partial n} \right) dS = - \int_{S_c^\epsilon} \left(\phi^I \frac{\partial \phi^{R,k}}{\partial n} - \phi^{R,k} \frac{\partial \phi^I}{\partial n} \right) dS \tag{5}$$

where it has been implicitly used that the contribution from the integrals on the free surface and the seabed are zero.

By substituting the expressions of the incident and radiated potentials (1) into (4), with the surface integral over S_c^ϵ evaluated using Bessel and Hankel identities, the latter becomes:

$$\mathbf{G}_m^k = 4 \frac{\rho c_g \omega^2}{g k_0} (-1)^m R_{-m}^k \tag{6}$$

where c_g is the group velocity.

Figure 2 shows the progressive terms of the FTM in surge ($\mathbf{G}_m^1$) for a floating vertical cylinder computed both by means of the BEM code NEMOH using its new capability to solve the diffraction problem for partial cylindrical waves and by means of the RC (computed with the same solver implementing the methodology by Goo and Yoshida (1990)), using the identity (6). The excellent agreement between the two calculation approaches provides a first numerical check of the extended Haskind's relation (6).

2.2 Array

The Haskind's relation can be generalized for an array composed of N_b bodies (Falnes, 1980):

$$F_{ex,i}^{k_i} = -i\rho\omega \int_{S_\infty} \left(\phi^I \frac{\partial \Phi_i^{R,k_i}}{\partial n} - \Phi_i^{R,k_i} \frac{\partial \phi^I}{\partial n} \right) dS \tag{7}$$

where Φ_i^{R,k_i} is the potential of a wave radiated by body i in a mode of motion k_i and scattered by all the neighbouring bodies j at rest.

The representation of the incident and radiated potentials in expression (7) relating the excitation force with the far-field potentials follows as in (1). However, as the body is no longer in isolation, the radiated potential of a body i moving in mode of motion k_i can be written as a sum of two contributions:

$$\Phi_i^{R,k_i} = \phi_i^{R,k_i} + \sum_{j=1}^{N_b} \phi_j^{S,k_i} \tag{8}$$

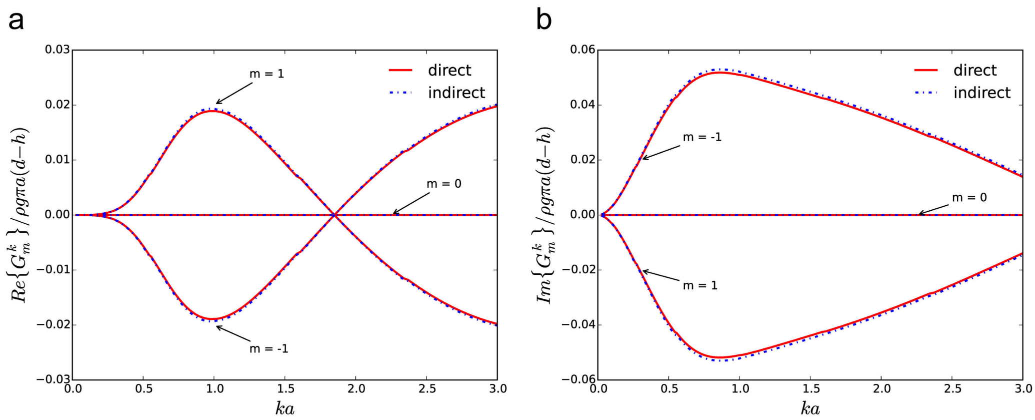


Figure 2: **Real (a) and Imaginary (b) parts of the FTM progressive terms in the surge (k=1) degree of freedom for a cylinder of radius a, draft $(d-h=2a)$ in a water depth $(d=4a)$. The solid line (direct) corresponds to the direct calculation of the FTM using NEMOH. The dotted line (indirect) is calculated from the RC computed with our BEM solver and by using the right-hand side of equation (6).**

where Φ_i^{R,k_i} is the total radiated potential, ϕ_i^{R,k_i} is the radiated potential by body i in motion mode k_i as if it was isolated and ϕ_j^{S,k_i} is the scattered potential by a body j in the array due to the wave radiated by body i moving in mode of motion k_i.

As the potentials are expressed with respect to the local cylindrical reference system centered at each body, we apply a coordinate transformation to express all potentials with respect to the local reference system of body i. We make use of the multipole expansion matrix $\mathbf{M}_{ij}$ from Graff's addition theorem which expresses the scattered potential of body i around the origin of the j^{th} coordinate system (Kashiwagi, 2000):

$$\Upsilon_i^S(r_i,\theta_i) = \mathbf{M}_{ij}\,\Upsilon_j^S(r_j,\theta_j) \quad \text{with} \quad (\mathbf{M}_{ij})_{nn}^{mq} = J_{m-q}(k_0 L_{ij})\,e^{i(m-q)\alpha_{ij}}; \quad n=0 \tag{9}$$

where $\Upsilon^S = H_m(k_0 r)\,e^{im\theta}$.

By applying (9) and using (1), equation (8) can be expressed as:

$$\Phi_i^{R,k_i}(r_i,\theta_i,z_i) = f(z)\sum_{m=-\infty}^{\infty}(\mathcal{R}_i^{k_i})_m H_m(k_0 r_i)e^{im\theta_i} \quad \text{with} \quad (\mathcal{R}_i^{k_i})_m = \big(R_i^{k_i} + \sum_{j=1}^{N_b}[\mathbf{M}_{ji}]^T A_j^{k_i}\big)_m \tag{10}$$

where $\mathcal{R}_i^{k_i}$ is the vector of cylindrical coefficients expressing the radiated wave by body i in a motion mode k and including all the scattered waves by all the bodies in the array.

Finally, expression (7) is evaluated as:

$$\boxed{F_{ex,i}^{k_i} = \sum_{m=-\infty}^{\infty}\Big(\mathbf{G}_m^{k_i} + \tilde{\mathbf{G}}_m^{k_i}\Big)\,a_m^{I,i}} \quad \text{with} \quad \tilde{\mathbf{G}}_m^{k_i} = 4\frac{\rho c_g \omega^2}{g k_0}(-1)^m\big(\sum_{j=1}^{N_b}[\mathbf{M}_{ji}]^T A_j^{k_i}\big)_m \tag{11}$$

Figure 3 shows the variation of the excitation force with wave number in surge and heave for cylinders 1-2 in the array shown in Figure 1b and for an incident wave propagating with $\beta = \pi/4$ direction. Results obtained by Siddorn and Eatock Taylor (2008) are shown and compared to our two different computations. The first one using relationship (11) and the second one with a direct IT calculation whose hydrodynamic operators have been obtained using NEMOH. A very good match between them is found with a slight overestimation of the peaks for surge, mainly at low wave numbers ($ka < 1$). By means of expression (11), the separate contribution from the isolated body and the hydrodynamic interactions to the total excitation force have been computed and are also shown in the plot. Only the absolute value is displayed here, so the curves cannot be directly summed. A near-trapped mode at $ka = 1.66$ can be observed.

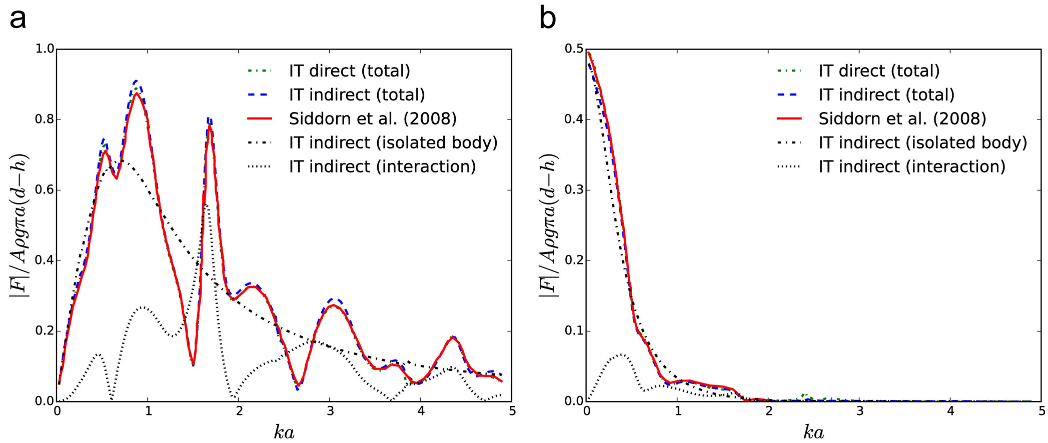


Figure 3: **Non-dimensional excitation forces in Surge (a) and Heave (b) for the cylinders 1-2 in the array shown in Figure 1b from an incident plane wave with propagation direction ($\beta = \pi/4$) and amplitude A . The solid line reproduces the results by Siddorn and Eatock Taylor (2008); the dotted green line has been computed with the IT by Kagemoto and Yue (1986) using NEMOH to compute the required hydrodynamic operators and the dotted blue line by means of (11). The black dotted lines ($- \cdot -\cdot$) and ($\cdot\cdot\cdot$) correspond respectively to the contribution to the total excitation force from the isolated body and from the hydrodynamic interactions with the neighbours.**

3 CONCLUSIONS

By applying the Haskind's relation to the representation of the potential using cylindrical harmonics we were able to derive a new simple relationship between the Force Transfer Matrix and the Radiation Characteristics that we have called *extended* Haskind relation. This removes the need to solve any radiation problem, with the solution to the diffraction problem being known from the computation of the DTM, and the numerical integration of the source strengths over the wetted surface of the body to calculate the RC as required by the methodology of Goo and Yoshida (1990). In addition, we extended the identity for the case of arrays and provided an expression which can be used to validate an implementation of the interaction theory by Kagemoto and Yue (1986).

ACKNOWLEDGEMENTS

The research leading to these results is part of the OceaNET project, which has received funding from the European Union's Seventh Framework Programme for research, technological development and demonstration under grant agreement no 607656.

REFERENCES

S.K. Chakrabarti. Response due to moored multiple structure interaction. *Marine Structures*, 14(1-2):231–258, January 2001. ISSN 09518339.

Johannes Falnes. Radiation impedance matrix and optimum power absorption for interacting oscillators in surface waves. *Applied Ocean Research*, 2(2):75–80, 1980.

Ja-Sam Goo and Koichiro Yoshida. A Numerical Method for Huge Semisubmersible Responses in Waves. *Society of Naval Architects and Marine Engineers*, 98:365–387, 1990.

Hiroshi Kagemoto and Dick K. P. Yue. Interactions among multiple three-dimensional bodies in water waves: an exact algebraic method. *Journal of Fluid Mechanics*, 166 (-1):189, April 1986.

M. Kashiwagi. Hydrodynamic interactions among a great number of columns supporting a very large flexible structure. *Journal of Fluids and Structures*, 14:1013–1034, 2000.

J. Cameron McNatt, Vengatesan Venugopal, and David Forehand. A novel method for deriving the diffraction transfer matrix and its application to multi-body interactions in water waves. *Ocean Engineering*, 94:173–185, January 2015. ISSN 00298018.

J.N. Newman. The Exciting Forces on Fixed Bodies in Waves. *Journal of Ship Research*, 6(10), 1962.

Malte A. Peter and Michael H. Meylan. Infinite-depth interaction theory for arbitrary floating bodies applied to wave forcing of ice floes. *Journal of Fluid Mechanics*, 500:145–167, January 2004. ISSN 0022-1120.

P. Siddorn and R. Eatock Taylor. Diffraction and independent radiation by an array of floating cylinders. *Ocean Engineering*, 35(13):1289–1303, September 2008. ISSN 00298018.