# Water Wave Interaction With Several Non-Circular Vertical Cylinders

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## 1 INTRODUCTION

We consider the linear problem of water waves scattering by $N$ vertical cylinders with non-circular cross sections extending from the sea bottom to the free surface in water of finite depth $h$. It is assumed that a plane wavetrain incident from $-\infty$ and propagating at an angle $\alpha$ to the positive $x$-direction towards the vertical cylinders whose cross sections are described by the equations, $r_j = R_j[1 + \varepsilon_j f_j(\theta_j)]$, with $\varepsilon_j \ll 1$, $j = 1,2, \dots, N$. The functions $f_j(\theta_j)$ describes the deviation of the shape of the cylinder $j$ from the circular one with $(r_j, \theta_j)$ denoting the polar coordinates placed at the center of cylinder $j$. The problem of wave scattering by a nearly circular cylinder was formulated in (Mei et al., 2005) and a fifth-order asymptotic solution of the problem has been obtained for the cylinders with elliptic, quasi- elliptic, square cross sections and cylinders with cosine type radial perturbations by the authors. (Dişibüyük, Korobkin, Yilmaz, 2016). Several numerical methods are available for the calculation of diffraction by multiple cylinders and they are discussed by Mei (1978). For semi-analytical solutions for interaction of vertical circular cylinders in an array see, Spring and Monkmeyer (1974), Ohkusu (1974), Kagemoto and Yue (1986), Linton and Evans (1990). The interaction of waves by arrays of elliptic cylinders are given by Chatjigeorgiou and Mavrakos, (2010). In this study, the asymptotic method for a single cylinder of arbitrary cross section (Dişibüyük, Korobkin, Yilmaz, 2016) and the iterative method for multiple circular cylinders (Yılmaz, 2004) are combined to solve the interaction problem for arbitrary number of cylinders with arbitrary cross sections. Wave forces acting on two elliptic cylinders are presented.

## 2 MATHEMATICAL FORMULATION OF THE PROBLEM

The linear boundary problem is formulated with respect to the velocity potential $\Phi(r, \theta, z, t)$
$\Phi(r, \theta, z, t) = R\left\{\frac{gA}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\phi(r,\theta)e^{-i\omega t}\right\}$,

where $\phi$ satisfies the Helmholtz equation $(\nabla^2 + k^2)\phi = 0$ in the flow region, $A$ is the incident wave amplitude, $k = \frac{2\pi}{\lambda}$ is the wave number, $\lambda$ is the incident wave length, $\omega$ is the wave frequency related to the wave number $k$ by the dispersion relation $\omega^2 = gk \tanh kh$, where $g$ is the gravitational acceleration. $N + 1$ coordinate systems are used: $(r, \theta, z)$ with the origin at the free surface and the $z$-axis upward and local coordinates $(r_j, \theta_j, z_j)$, $j = 1, \dots, N$ centered at the origin of each cylinder $(x_j, y_j)$. $L_{ji}$ is the distance between the center of the cylinder $j$ and $i$, (see Fig. 1).

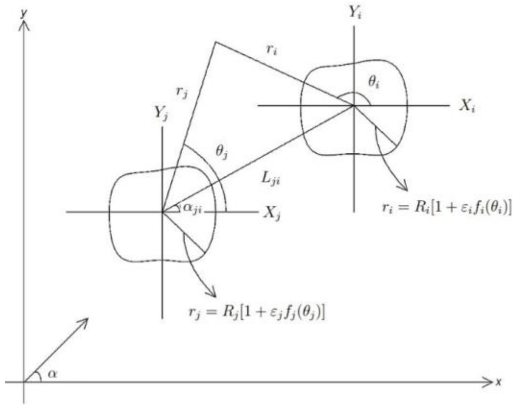


**Fig. 1: Basic configuration and coordinate systems.**

The basic idea of the interaction is that for each cylinder $j$, the waves arriving from other cylinders are treated as incident wave. Hence the total wave potential for cylinder $j$ is

$$\phi_j^{(p)}(r_j,\theta_j) = \sum_{m=0}^{\infty}\left[v_{jm}^{(p)}\cos(m\theta_j) + \tilde{v}_{jm}^{(p)}\sin(m\theta_j)\right]J_m(kr_j) + \sum_{m=0}^{\infty}\left[b_{jm}^{(p)}\cos(m\theta_j) + c_{jm}^{(p)}\sin(m\theta_j)\right]H_m^{(1)}(kr_j),\ \ p = 1,2,\dots,\ \ j = 1,\dots,N. \qquad (1)$$

where $p$ defines the number of iteration and the first summation in (1) is the sum of the diffracted waves from other cylinders which are transformed to the coordinates $(r_j, \theta_j)$ by the addition theorem of Bessel functions and the incoming wave from infinity. The second summation in (1) represents the diffraction of the total incident wave from cylinder $j$. For the first iteration ($p = 1$) and the first cylinder ($j = 1$), $v_{1m}^{(1)} = \epsilon_m i^m$, $\tilde{v}_{1m}^{(1)} = 0$ where $\epsilon_m$ is the Neumann symbol, $\epsilon_0 = 1$, $\epsilon_m = 2$ for $m \geq 1$. The unknown coefficients $b_{jm}^{(p)}$, $c_{jm}^{(p)}$ are found from the boundary condition:

$$\frac{\partial\phi_j^{(p)}}{\partial n_j} = 0 \text{ on } r_j = R_j[1 + \varepsilon_j f_j(\theta_j)],\ \ j = 1,\dots,N.$$

where $\boldsymbol{n}_j$ is the unit normal vector on the cylinder $j$ and $\phi_j$ is the velocity potential in the local coordinates of cylinder $j$. This boundary condition can be written as

$$\frac{\partial\phi_j^{(p)}}{\partial r_j}\left(R_j\left[1 + \varepsilon_j f_j(\theta_j)\right],\theta_j\right) - \frac{\varepsilon_j f_j'(\theta_j)}{R_j\left[1+\varepsilon_j f_j(\theta_j)\right]^2}\frac{\partial\phi_j^{(p)}}{\partial\theta_j}\left(R_j\left[1 + \varepsilon_j f_j(\theta_j)\right],\theta_j\right) = 0,\ j = 1,\dots,N, \qquad (2)$$

where $0 < \theta_j < 2\pi$. We approximate the boundary condition (2) up to $O(\varepsilon^5)$ using the Taylor expansions at $r_j = R_j$ , $j = 1,\dots,N$ and substituting the fifth order asymptotic expansion of the potential $\phi_j$

$$\phi_j^{(p)}(r_j,\theta_j) = \phi_{j0}^{(p)}(r_j,\theta_j) + \varepsilon\phi_{j1}^{(p)}(r_j,\theta_j) + \varepsilon^2\phi_{j2}^{(p)}(r_j,\theta_j) + \varepsilon^3\phi_{j3}^{(p)}(r_j,\theta_j) + \varepsilon^4\phi_{j4}^{(p)}(r_j,\theta_j) + O(\varepsilon^5), \qquad (3)$$

into the boundary condition (2). We obtain

$\phi_{j0}^{(p)}(R_j, \theta_j) = 0,$

$\phi_{j1}^{(p)}(R_j, \theta_j) = \frac{1}{{R_j}^2} f_j'(\theta_j)\phi_{j0,\theta_j}^{(p)}(R_j, \theta_j) - f_j(\theta_j)\phi_{j0,r_j r_j}^{(p)}(R_j, \theta_j),$

at the order of $\varepsilon^0$ and $\varepsilon^1$ respectively. The boundary conditions for higher orders of $\varepsilon$ (up to $\varepsilon^5$) are obtained similarly. It is clear that $\phi_{j0}^{(p)}(r_j, \theta_j)$ is the velocity potential of the diffraction problem for the circular cylinder $r_j = R_j$ (see MacCamy and Fuchs, 1954). The most general representation of $\phi_{jn}^{(p)}(r_j, \theta_j)$, $n = 1,2,3,4$ in (3), which satisfy the radiation condition at infinity, $\phi_{jn}^{(p)} \to 0$ as $r \to \infty$, is

$$\phi_{jn}^{(p)}(r_j, \theta_j) = \sum_{m=0}^{\infty} \left[B_{jmn}^{(p)} \cos[m(\theta_j - \alpha)] + C_{jmn}^{(p)} \sin[m(\theta_j - \alpha)]\right] H_m^{(1)}(kr_j), \quad j = 1, \dots, N.$$

By expanding in a Fourier series and using the boundary conditions (4), (5) and the other conditions corresponding to higher orders of $\varepsilon$ the unknown coefficients $B_{jmn}^{(p)}$ and $C_{jmn}^{(p)}$, $j = 1, \dots, N$, $n = 1,2,3,4$, $m = 0,1,2, \dots$ are determined.

Now, the unknown coefficients $b_{jm}^{(p)}$ and $c_{jm}^{(p)}$, $m = 0,1,2, \dots,$ , $j = 1, \dots, N$ in (1) are given by

$b_{jm}^{(p)} = -v_{mj}^{(p)} \frac{J_m'(kR_j)}{H_m^{(1)\prime}(kR_j)} + \varepsilon B_{jm1}^{(p)} + \varepsilon^2 B_{jm2}^{(p)} + \varepsilon^3 B_{jm3}^{(p)} + \varepsilon^4 B_{jm4}^{(p)},$

$c_{jm}^{(p)} = -\tilde{v}_{mj}^{(p)} \frac{J_m'(kR_j)}{H_m^{(1)\prime}(kR_j)} + \varepsilon C_{jm1}^{(p)} + \varepsilon^2 C_{jm2}^{(p)} + \varepsilon^3 C_{jm3}^{(p)} + \varepsilon^4 C_{jm4}^{(p)}.$

This process of iteration is continued until the desired accuracy $\left|\phi_j^{(p+1)} - \phi_j^{(p)}\right| < \delta$, where $\delta$ is a small number, $j = 1, \dots, N$.

The non-dimensional $x_j$ and $y_j$ components of the hydrodynamic force due to the fluid motion on the cylinder $j$ are given by

$$\tilde{F}_{x_j} = \frac{-iR_j \tanh(kh)}{ka_j^2} \int_0^{2\pi} \phi_j^{(p)}(R_j[1 + \varepsilon_j f_j(\theta_j)], \theta_j)[\varepsilon_j f_j'(\theta_j) \sin\theta_j + [1 + \varepsilon_j f_j(\theta_j)] \cos\theta_j] d\theta_j,$$

$$\tilde{F}_{y_j} = \frac{-iR \tanh(kh)}{ka_j^2} \int_0^{2\pi} \phi_j^{(p)}(R_j[1 + \varepsilon_j f_j(\theta_j)], \theta_j)[-\varepsilon_j f_j'(\theta_j) \cos\theta_j + [1 + \varepsilon_j f_j(\theta_j)] \sin\theta_j] d\theta_j.$$

$j = 1, \dots, N$, which are scaled by $\rho g A a_j^2$, where $a_j$ is a characteristic dimension of the $j$-th cylinder cross section.

## 3 RESULTS AND CONCLUSIONS

As an example an arrangement of two eliptical cylinders is considered with the same dimensions as in the paper of (Chatjigeorgiou and Mavrakos, 2010). The semi-minor and semi-major axis of the elliptic cylinders are $b$ and $a$ respectively. The following ratios are used: $b/a = 0.4$, $h/a = 0.8$ and $L_{ji}/a = 2$. The center of the cylinders are at (0,0) and $(0,2a)$. The equation $r_j = aF_j(\theta_j)$, $j = 1,2$ describe the ellipse in the polar coordinates $(r_j, \theta_j)$ whose origin is at its center, where $F_j(\theta_j) = \sqrt{1-e^2}/(1-e\cos\theta_j)^2$, $e = \sqrt{1-(b^2/a^2)}$, $0 < e < 1$, is the eccentricity of the ellipse. The Fourier coefficients of the function $F_j(\theta_j)$, $0 \leq \theta_j \leq 2\pi$, are determined and then converted to the corresponding Fourier series into the form $r_j = R_j[1+\varepsilon_j f_j(\theta_j)]$ identifying values of $R_j$, $\varepsilon_j$, and the function $f_j(\theta_j)$.

The asymptotic method for a single cylinder of arbitrary cross section (Dişibüyük, Korobkin, Yilmaz, 2016) and the iterative method for multiple circular cylinders (Yılmaz, 2004) are combined to solve the interaction problem for arbitrary number of cylinders with arbitrary cross sections. For the interaction of two elliptic cylinders, our results are compared with the results of Chatjigeorgiou and Mavrakos, (2010) who used the expansion of the exact expressions for the forces which are given by the Mathieu functions. The present asymptotic approach provides a good approximation for the forces exerted on the elliptic cylinders to the different incident wave values. (see Fig. 2).

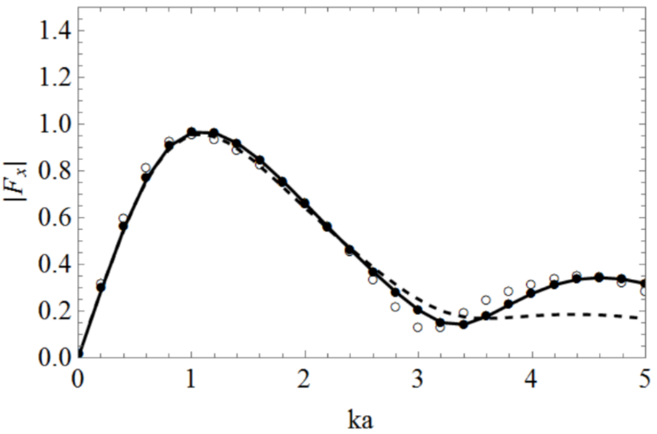


**Fig. 2: The $x$-component of the non-dimensionlized force on elliptic cylinders for $\alpha = 0^{\circ}$. The solution by (Chatjigeorgiou and Mavrakos, 2010) for cylinders 1 and 2 (solid line), the present method with one iteration ($p = 1$) for cylinder 1 (• markers) and cylinder 2 (◦ markers). Dashed line is solution for one elliptic cylinder by (Chatjigeorgiou and Mavrakos, 2010).**

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