Higher-order asymptotic expansions of Fourier integral with two nearly coincident saddle points

Yuzhi Dai[1], **Xiaobo Chen**[2]
[1]Shipbuilding Engineering College, Harbin Institute of Technology (Weihai), Weihai, China
[2]Deepwater Technology Research Centre, BV, Singapore
email: dyzbs@hitwh.edu.cn

Further to the uniform asymptotic expansion of Fourier integral with large parameter and two nearly coincident saddle points presented at the workshop by Dai and Chen (2013), we have also obtained the higher-order non-uniform asymptotic expansions, which can be regarded as the generalization of Kelvin and Peters's result. The non-uniform asymptotic expansion gives important information about the physical behavior of this Fourier integral in different regions. Uniform asymptotic expansions can be used as the basis of their numerical approximations. Combination of these two results provides the complete solution to the problem proposed by Lord Kelvin.

1 Basic formulations

The basis to develop non-uniform asymptotic expansion of an integral of Fourier type is given by Olver (1974) :

$$\int_0^\infty u^{\alpha-1} e^{itu} du = e^{i\pi\alpha/2}\Gamma(\alpha)t^{-\alpha} \tag{1}$$

with the real parameter $t > 0$ and complex α satisfying that $\Re(\alpha) > 0$.

By using (1) introduced above, we can have following identities :

$$\int_{-\infty}^\infty u^{m-1} e^{ixu^n} du = \begin{cases} (2/n)\exp(i\Sigma\pi m/2n)\Gamma(m/n)|x|^{-m/n} & m \text{ odd and } n \text{ even}, \\ 0 & m \text{ even and } n \text{ even}, \\ (2/n)\cos(\Sigma\pi m/2n)\Gamma(m/n)|x|^{-m/n} & m \text{ odd and } n \text{ odd}, \\ i(2/n)\sin(\Sigma\pi m/2n)\Gamma(m/n)|x|^{-m/n} & m \text{ even and } n \text{ odd}, \end{cases} \tag{2}$$

in which x is real and $\Sigma = \mathrm{sign}(x)$, and m, n are positive integers.

Furthermore, we define a complex parameter z with $\Im z > 0$. By applying the Cauchy theorem, we have :

$$\int_{-\infty}^\infty u^{2n} e^{izu^2} du = e^{i\pi(2n+1)/4}\Gamma(n+1/2)z^{-(n+1/2)} \tag{3}$$

where the phase z is comprised in $(-\pi, \pi]$.

2 Higher-order asymptotic expansions

We consider the Fourier-type integral $I(\lambda, \theta)$ which is defined as :

$$I(\lambda,\theta) = \int_a^b f(k) e^{i\lambda\varphi(k,\theta)} dk \tag{4}$$

with a large parameter λ. In (4), the parameters λ and θ are real while the amplitude function $f(k)$ and phase function $\varphi(k, \theta)$ are real functions. Suppose that there exists a critical value of θ_c, two distinct real saddle points k_1 and k_2 comprising in (a, b), i.e., $a < k_1 < k_2 < b$ for $\theta < \theta_c$, and two complex conjugate saddle points for $\theta > \theta_c$. As $\theta \to \theta_c$, the two saddle points coincide at a single saddle point $k = k_c$ of order 2.

2.1 $\theta < \theta_c$

For large value of λ, the integral (4) can be approximated by the contribution near the saddle points

$$I(\lambda,\theta) \approx I_1 + I_2 \tag{5}$$

with the contribution of integration in the vicinity of k_1 and k_2 approximated by :

$$I_1 = \int_{k_1-\epsilon}^{k_1+\epsilon} f(k) e^{i\lambda\varphi(k,\theta)} dk, \qquad I_2 = \int_{k_2-\epsilon}^{k_2+\epsilon} f(k) e^{i\lambda\varphi(k,\theta)} dk \tag{6}$$

in which $0 < \epsilon \ll 1$. The first integral can be further estimated by introducing the Taylor development of the amplitude and phase functions :

$$
\begin{aligned}
I_1 = & \int_{k_1-\epsilon}^{k_1+\epsilon} \left[f(k_1) + f'(k_1)(k-k_1) + f''(k_1)(k-k_1)^2/2\right] \\
& \quad \times \exp\left\{i\lambda\left[\varphi(k_1,\theta) + \varphi''(k_1,\theta)(k-k_1)^2/2 + \varphi'''(k_1,\theta)(k-k_1)^3/6 + \varphi''''(k_1,\theta)(k-k_1)^4/24\right]\right\} dk \\
& \approx \int_{-\infty}^{\infty} \exp\left\{i\lambda\left[\varphi(k_1,\theta) + \varphi''(k_1,\theta)k^2/2\right]\right\} \\
& \quad \times \left\{f(k_1) + f''(k_1)k^2/2 + i\lambda[f'(k_1)\varphi'''(k_1,\theta)/6 + f(k_1)\varphi''''(k_1,\theta)/24] - \lambda^2 f(k_1)[\varphi'''(k_1,\theta)]^2 k^6/72\right\} dk \\
& \approx f(k_1)\exp\left[i\lambda\varphi(k_1,\theta) + \mathrm{sgn}(\varphi_1'')i\pi/4\right]\sqrt{2\pi/[\lambda|\varphi''(k_1,\theta)|]} \\
& \quad \times \left\{1 + \frac{i}{2\lambda}\left[\frac{f''(k_1)}{f(k_1)\varphi''(k_1,\theta)} - \frac{\varphi''''(k_1,\theta)}{4[\varphi''(k_1,\theta)]^2} - \frac{f'(k_1)\varphi'''(k_1,\theta)}{f(k_1)[\varphi''(k_1,\theta)]^2} + \frac{5[\varphi'''(k_1,\theta)]^2}{12[\varphi''(k_1,\theta)]^3}\right]\right\}
\end{aligned} \tag{7}
$$

where $\mathrm{sgn}(\varphi_1'') = \varphi''(k_1,\theta)/|\varphi''(k_1,\theta)|$. The prime in above and hereafter denotes differentiation with respect to k. In the same way, we can obtain the approximation of I_2 so that

$$
\begin{aligned}
I(\lambda,\theta) \approx & \sum_{j=1}^{2} f(k_j)\exp\left[i\lambda\varphi(k_j,\theta) + \mathrm{sgn}(\varphi_j'')i\pi/4\right]\sqrt{2\pi/[\lambda|\varphi''(k_1,\theta)|]} \\
& \times \left\{1 + \frac{i}{2\lambda}\left[\frac{f''(k_j)}{f(k_j)\varphi''(k_j,\theta)} - \frac{\varphi''''(k_j,\theta)}{4[\varphi''(k_j,\theta)]^2} - \frac{f'(k_j)\varphi'''(k_j,\theta)}{f(k_j)[\varphi''(k_j,\theta)]^2} + \frac{5[\varphi'''(k_j,\theta)]^2}{12[\varphi''(k_j,\theta)]^3}\right]\right\}
\end{aligned} \tag{8}
$$

2.2 $\theta > \theta_c$

Two saddle points k_1 and k_2 are complex conjugate. Suppose that the imaginary part of $\varphi(k_1,\theta)$ is positive. The contribution from the saddle point k_2 is then negligible comparing to that from the saddle point k_1 which is obtained in the same way as (7) :

$$
\begin{aligned}
I(\lambda,\theta) \approx & f(k_1)\exp\left[i\lambda\varphi(k_1,\theta) + i\pi/4\right]\sqrt{2\pi/[\lambda\varphi''(k_1,\theta)]} \\
& \times \left\{1 + \frac{i}{2\lambda}\left[\frac{f''(k_1)}{f(k_1)\varphi''(k_1,\theta)} - \frac{\varphi''''(k_1,\theta)}{4[\varphi''(k_1,\theta)]^2} - \frac{f'(k_1)\varphi'''(k_1,\theta)}{f(k_1)[\varphi''(k_1,\theta)]^2} + \frac{5[\varphi'''(k_1,\theta)]^2}{12[\varphi''(k_1,\theta)]^3}\right]\right\}
\end{aligned} \tag{9}
$$

Note that $\varphi(k,\theta)$ and its derivatives at $k = k_1$ are complex for $\theta > \theta_c$, while they are real in the case when $\theta < \theta_c$.

2.3 $\theta = \theta_c$

Two saddles points k_1 and k_2 coalesce at $k = k_c$ at which $\varphi'(k_c,\theta_c) = 0 = \varphi''(k_c,\theta_c)$. Then we have :

$$
\begin{aligned}
I(\lambda,\theta_c) \approx & \int_{k_c-\epsilon}^{k_c+\epsilon} [f(k_c) + f'(k_c)(k-k_c)] \\
& \quad \exp\left\{i\lambda\left[\varphi(k_c,\theta_c) + \varphi'''(k_c,\theta_c)(k-k_c)^3/6 + \varphi''''(k_c,\theta_c)(k-k_c)^4/24\right]\right\} dk \\
& \approx f(k_c)\exp[i\lambda\varphi(k_c,\theta_c)] \int_{-\infty}^{\infty} \exp\left\{i\lambda\varphi'''(k_c,\theta_c)k^3/6\right\}[1 + i\lambda\varphi''''(k_c,\theta_c)/24]\, dk \\
& \quad + f'(k_c)\exp[i\lambda\varphi(k_c,\theta_c)] \int_{-\infty}^{\infty} \exp\left\{i\lambda\varphi'''(k_c,\theta_c)k^3/6\right\} k\, dk \\
& = \frac{\sqrt{3}}{3} f(k_c)\exp[i\lambda\varphi(k_c,\theta_c)]\Gamma(1/3)\left[\frac{6}{\lambda|\varphi'''(k_c,\theta_c)|}\right]^{1/3} \\
& \quad \times \left\{1 + i\,\mathrm{sgn}(\varphi_c'''(k_c,\theta_c))\frac{f'(k_c)\Gamma(2/3)}{f(k_c)\Gamma(1/3)}\left[\frac{6}{\lambda|\varphi'''(k_c,\theta_c)|}\right]^{1/3} - i\frac{\varphi''''(k_c,\theta_c)\Gamma(2/3)}{36^{1/3}\Gamma(1/3)}\left[\frac{1}{\lambda|\varphi'''(k_c,\theta_c)|^4}\right]^{1/3}\right\}
\end{aligned} \tag{10}
$$

2.4 $\theta \approx \theta_c$

When θ is sufficiently close to θ_c, $k_{1,2} \to k_c$ and $\varphi''(k_{1,2},\theta) \to 0$. In this case, (8) and (9) cannot give a good asymptotic values of $I(\lambda,\theta)$. In fact, there exists a real value of $k = k_s \neq k_c$ for $\theta \approx \theta_c$ defined by the relation :

$$\varphi''(k_s,\theta) = 0 \tag{11}$$

Thus we can develop :

$$
\begin{aligned}
I(\lambda,\theta) \approx & \int_{k_s-\epsilon}^{k_s+\epsilon} [f(k_s) + f'(k_s)(k-k_s)] \\
& \qquad \exp\left\{ i\lambda \left[\varphi(k_s,\theta) + \varphi'(k_s,\theta)(k-k_s) + \varphi'''(k_s,\theta)(k-k_s)^3/6 + \varphi''''(k_s,\theta)(k-k_s)^4/24 \right] \right\} dk \\
& = 2\pi f(k_s) \exp\left[i\lambda\varphi(k_s,\theta)\right] \mathrm{Ai}(X) \left[\frac{2}{\lambda|\varphi'''(k_s,\theta)|}\right]^{1/3} \\
& \qquad - i2\pi \mathrm{sgn}(\varphi_s''') f'(k_s) \exp\left[i\lambda\varphi(k_s,\theta)\right] \mathrm{Ai}'(X) \left[\frac{2}{\lambda|\varphi'''(k_s,\theta)|}\right]^{2/3} \\
& \qquad + i\pi\lambda[\varphi''''(k_s,\theta)/12] f(k_s) \exp\left[i\lambda\varphi(k_s,\theta)\right] \mathrm{Ai}''''(X) \left[\frac{2}{\lambda|\varphi'''(k_s,\theta)|}\right]^{5/3}
\end{aligned}
\tag{12}
$$

where $X = [2\lambda^2/\varphi'''(k_s,\theta)]^{1/3}\varphi'(k_s,\theta)$. It can be shown that (12) reduces to (10) when $\theta = \theta_c$.

3 Discussions

Application of the present results to Kelvin waves and comparison with Peters's result will be presented at the workshop.

References

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