Revisit of Cauchy-Poisson Problem in unsteady water wave problem

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INTRODUCTION

The traditional water wave problems were mostly solved under the assumption of harmonic oscillation in time. This assumption can provide a lot of advantages in the analysis. However, there are situations when the initial value problem approaches are necessary. The most famous one must be Cauchy-Poisson problem in water waves. Since this approach needs initial condition for the initial elevation and initial value of the potential, the delta function was adopted for the initial impulse. The present research used the initial elevation based on video recording. Therefore the present approach is considered to be based on physics. The video recording needs narrow field of view. Therefore considering the scale of generated waves the inclusion of surface tension effect is necessary. The mathematical formulation included surface tension. The numerical results are compared with those of experiment.

MATHEMATICAL FORMULATION

The axisymmetric wave field is considered. Therefore velocity potential is a function of r, z and t. The governing equation can be written in a form under the assumption of inviscid and incompressible fluid, and irrotational flow

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{1}$$

Two boundary conditions are needed for free surface.
The linearized kinematic free surface boundary condition is given

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{on } z=0 \tag{2}$$

The linearized dynamic free surface boundary condition including surface tension effect is

$$\frac{\partial \phi}{\partial t} + g\eta - \frac{T}{\rho}\left(\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r}\right) = 0 \quad \text{on } z=0 \tag{3}$$

Since we are going to solve initial value problem we are supposed to specify initial conditions for free surface elevation and potential. The integral transformation is employed here. The zeroth order Hankel and Laplace transformation is applied to velocity potential as

$$\bar{\phi}(k,z,s) = \int_0^\infty e^{-st} dt \int_0^\infty r J_0(kr)\phi(r,z,t)dr \tag{4}$$

Its inverse can be written

$$\phi(r,z,t) = \frac{1}{2\pi i}\int_\Gamma e^{st} ds \int_0^\infty k J_0(kr)\bar{\phi}(k,z,s)dk \tag{5}$$

After the transformation Eq. 1 becomes (Dautray, 1985)

$$-k^2\bar{\phi}(k,z,s)+\frac{d^2\bar{\phi}(k,z,s)}{dz^2}=0 \tag{6}$$

Assume deep water condition. Then

$$\bar{\phi}(k,z,s)=A(k,s)e^{kz} \tag{7}$$

The linearized kinematic free surface boundary condition Eq. 2 is transformed

$$s\bar{\eta}(k,s)-\bar{\eta}(k,0)=kA(k,s) \text{ on } z=0 \tag{8}$$

The linearized dynamic boundary condition turns into

$$sA(k,s)-\bar{\phi}(k,0,0)+g\bar{\eta}(k,s)+\frac{T}{\rho}k^2\bar{\eta}(k,s)=0 \text{ on } z=0 \tag{9}$$

We can have the expression for the elevation by solving Eqs. 8~9 with an additional assumption of

$$\bar{\phi}(k,0,0)=0$$

$$\bar{\eta}(k,s)=\frac{1}{s^2+gk+\frac{Tk^3}{\rho}}s\bar{\eta}(k,0) \text{ on } z=0 \tag{10}$$

Inversion of Hankel Laplace transformation leads to the free surface elevation

$$\eta(r,t)=\int_0^\infty kJ_0(kr)\bar{\eta}(k,0)\cos\sqrt{gk+\frac{Tk^3}{\rho}}t\,dk \tag{11}$$

If we can get the initial displacement as

$$\eta(r,0)=de^{-(\frac{r}{a})^2}\left\{1-(\frac{r}{a})^2\right\} \tag{12}$$

The Hankel transformation applied to Eq. 12 will give us (Debnath, 1994, Miles, 1998)

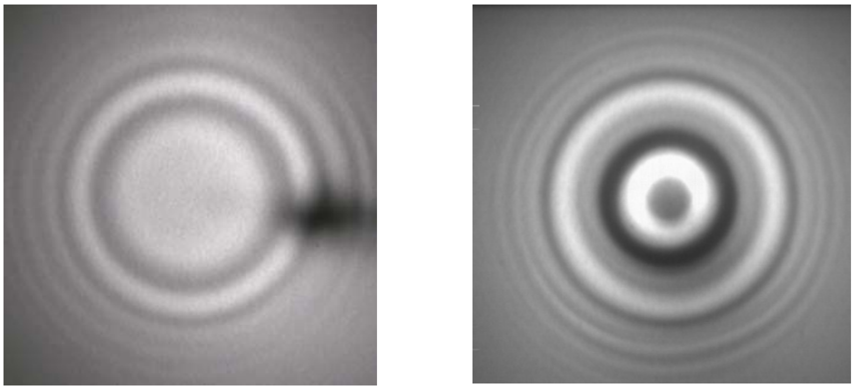
$$\bar{\eta}(k,0)=\frac{1}{8}da^2(ka)^2e^{-\frac{1}{4}(ka)^2} \tag{13}$$

Substitution of Eq. 13 into Eq. 11 yields solution for free surface elevation

$$\eta(r,t)=\frac{1}{8}da^4\int_0^\infty k^3J_0(kr)e^{-\frac{1}{4}(ka)^2}\cos\sqrt{gk+\frac{Tk^3}{\rho}}t\,dt \tag{14}$$

EXPERIMENT

The experiment was carried out in two water tanks. Dimensions of each tank are 60mm×60mm×60 mm (L×B×H) and 300mm×300mm×300mm. Small tank is used to record video clips for initial elevation. Large tank is used to record video clips for wave field. All video clips in this study are recorded by high speed camera. This camera has selectable recording rates up to 64,000fps. Two cases was set to investigate Capillary dominant wave (Case 1) and Capillary – Gravity wave (Case 2). Water droplet was dropped to generate waves. Two different masses of water drops were tested repeatedly to examine the standard deviation of each cases. The average mass of droplet for capillary dominant wave is 14.4mg and the average mass of droplet for Capillary-Gravity wave is 68.4mg. Case 1 was carried out in experimental condition with drop height of 1cm and water depth of 5cm. Case 2 was carried in condition with drop height of 4cm and water depth of 5cm. Each cases satisfy deep water condition.

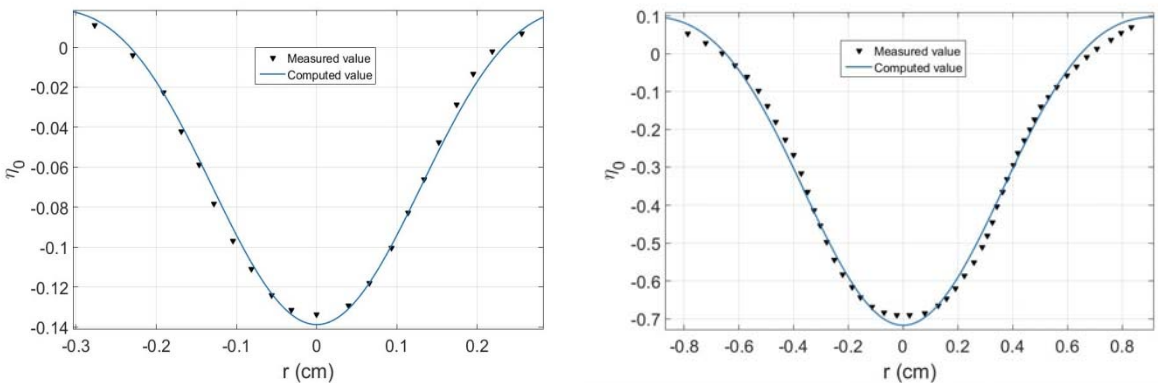


Capillary dominant wave   Capillary – Gravity wave

Fig. 1 Comparison of wave fields of Capillary dominant wave and Capillary-Gravity wave

RESULTS AND DISCUSSION

To obtain parameters ‘a’ and ‘d’ of new initial profile function, Eq. 12, the measured data from digitized initial elevation captured by high speed camera was curve fitted. Curve fitting results are shown in Fig. 2.



Case 1, Capillary dominant wave   Case 2, Capillary – Gravity wave

Fig. 2 Initial profiles for the free surface deformation

The symbols represent measured data from free surface deformation. The solid line denotes the curves fitted to Eq. 12. Using these values ‘a’ and ‘d’, we can obtain the solution for free surface elevation, Eq. 14. In Fig. 3, the solution of the classical Cauchy-Poisson problem which utilizes delta function as initial condition was compared with that of experimental result. As one can see, the initial profile function, Eq. 12 is a better choice to represent a real free surface in comparison with elevation solved by classical Cauchy-Poisson problem (Lamb, 1932).

Initial elevation η_0 in domain of k was plotted to show distribution of wave numbers. For Case 1, it has a larger dominant wave number based on the wave number of 3.6 rad/cm at the wavelength that separates the capillary wave and the gravity wave. For Case 2, it contains both capillary wave and gravity wave. The results are shown in Fig. 4.

Phase velocity was computed to compare with numerical value. For Case 1, Measured value of phase velocity is 25.9cm/s and numerical result is 25.3cm/s. The error between both values is 2.3%. For Case 2, measured value is 21.1cm/s and numerical value is 22.5cm/s. The error is 6.2%. These show experimental results agree quite closely with theoretical results.

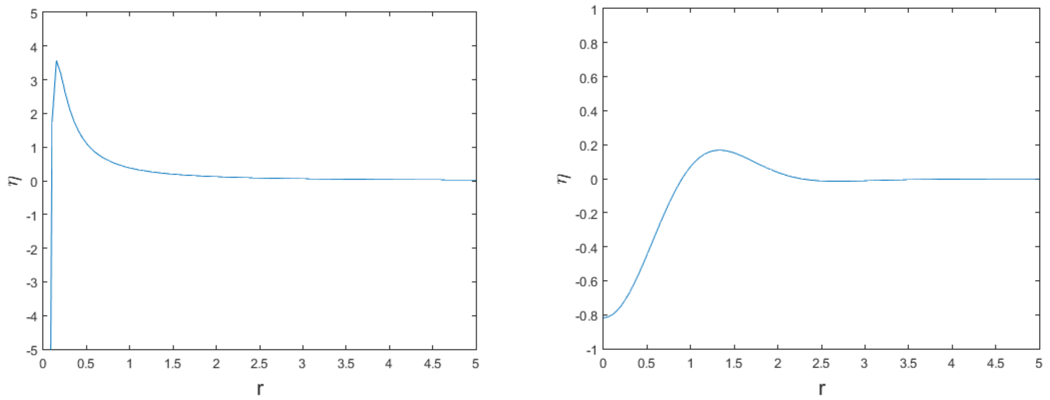


Fig. 3 Comparison of measurement and the solution of the classical Cauchy-Poisson Problem(t=0.5s)

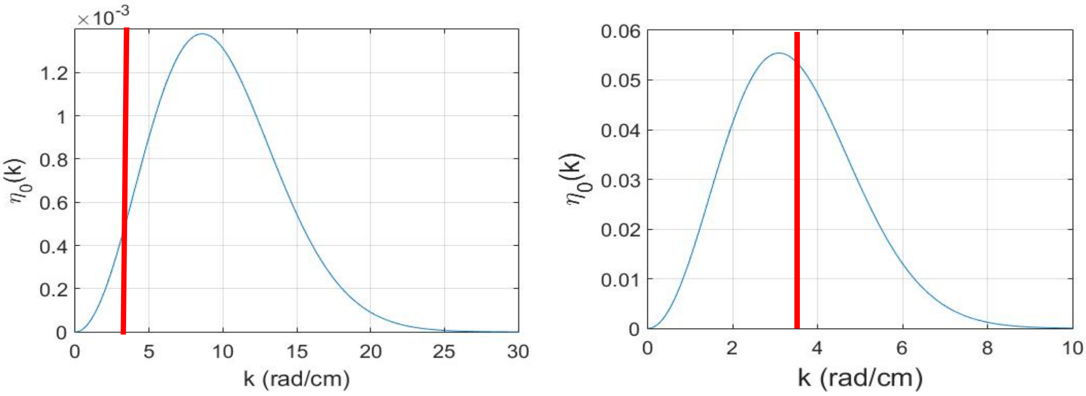


Fig. 4 Initial elevation in domain k for Case 1 (Left) and Case 2 (Right)

CONCLUDING REMARKS

The solution of the Cauchy-Poisson problem was investigated with measured initial profile. The proposed approach can be an alternative to the classical solution which utilizes delta function as initial condition. Sources of errors in comparison of theory and experimental results can be summarized as follows. Firstly, free surface boundary conditions are linearized. Secondly, Initial value of potential can't represent a real phenomenon completely. Thirdly initial condition for potential function without consideration of deformation was used.

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