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Coupling of Normal and Hypersingular Integral Equations in Wave-structure Interaction Problems

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Highlights:

- New boundary integral equation is established to improve the coefficient matrix of the linear system when applying boundary element method. Numerical tests for an inner domain problem was done and results are compared with analytical solution.
- Program for evaluating hypersingular integral is implemented and results are validated.
- An indirect method for evaluating free term coefficients is given for both inner domain and outer domain problems.

1. Introduction

Boundary Element Method(BEM) is widely used for studying wave-structure interaction problems. In most cases, the boundary integral equations(BIEs) are established using Green's second identity. By inserting source points at different node locations, the Laplace equation solving problem was turned into solving a linear system.

For an N-node mesh, the $N \times N$ coefficient matrix formed by discretizing the boundary using BIEs is not fully diagnoal dominant since Dirichlet(first type) boundary condition is given at the free water surface while Neumann(second type) boundary condition is given at the body surface. Kitagawa (1991) has proved that when N is large, stability of the solution will be deteriorated.

In this study, we proposed a diagonal dominant method by coupling the normal integral equation with a hypersingular integral equation. Numerical tests are performed to check the results of the new method.

2. Boundary Integral Equation

The boundary integral equation commonly used is derived from Green's second identity:

$$\lim_{\epsilon \to \infty} \iint_{S+S_\epsilon - e_\epsilon} \left(\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n} \phi(\mathbf{x}) - \frac{\partial \phi(\mathbf{x})}{\partial n} G \right) ds = 0 \tag{1}$$

After applying the limit to the integration over an infinitesimal sphere around a source point, Eqn(1) becomes:

$$\begin{aligned} \alpha \cdot \phi(\mathbf{y}) - \iint_{S_B} \phi(x) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n} ds(x) + \iint_{S_F} G(\mathbf{x},\mathbf{y}) \frac{\partial \Phi(x)}{\partial n} ds(x) \\ = - \iint_{S_B} G(\mathbf{x},\mathbf{y}) \frac{\partial \phi(x)}{\partial n} ds(x) + \iint_{S_F} \phi(x) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n} ds(x) \end{aligned} \tag{2}$$

α is related to solid angle. α's are much larger than off-diagonal elements for coefficient matrix of linear system and they are major contributors to the diagonal elements. However, this is only true for body surface nodes since $\phi(\mathbf{y})$ at the free water surface is given by boundary conditions and this term will be moved to right hand side of the equation. Hence, the new boundary integral equation is constructed by applying $\partial/\partial y_3$ operation to both sides of Eqn(1):

$$\lim_{\epsilon\to\infty} \iint_{S+S_\epsilon-e_\epsilon} \left(\frac{\partial^2 G(\mathbf{x},\mathbf{y})}{\partial y_3 \partial n}\phi(\mathbf{x}) - \frac{\partial \phi(\mathbf{x})}{\partial n}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial y_3} \right) ds = 0 \tag{3}$$

Guiggani(1992) gave an expanded form of Eqn(3)[1]:

$$\begin{aligned} C_{33}\cdot \phi_{,y_3}(\mathbf{x})|_{\mathbf{x}=\mathbf{y}} - \iint_{S_B} \Phi(x)\frac{\partial^2 G(\mathbf{x},\mathbf{y})}{\partial y_3 \partial n} ds(x) + \iint_{S_F} \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial y_3}\frac{\partial \Phi(x)}{\partial n} ds(x) \\ = - \iint_{S_B} \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial y_3}\frac{\partial \Phi(x)}{\partial n} ds(x) + \iint_{S_F} \Phi(x)\frac{\partial^2 G(\mathbf{x},\mathbf{y})}{\partial y_3 \partial n} ds(x) \\ - [a_3\cdot\phi(\mathbf{y}) + C_{32}\cdot\phi_{,y_2}(\mathbf{x})|_{\mathbf{x}=\mathbf{y}} + C_{31}\cdot\phi_{,y_1}(\mathbf{x})|_{\mathbf{x}=\mathbf{y}}] \end{aligned} \tag{4}$$

Similarly to Eqn(2), the coefficients $a_3, C_{31}, C_{32}, C_{33}$ arise from the integration over an infinitesimal sphere around a source point. The evaluation of these coefficients will be elaborated in later section. Besides, integration involved kernel $\partial^2 G/\partial y_3 \partial n$ is hypersingular and exposes more mathematical challenge for evaluation. If C_{33} is large enough, we can get a new diagonal dominant coefficient matrix for the linear system by combining Eqs(2) and (4).

3. Evaluation of Hypersingular Integral

Evaluation of hypersingular integral in Eqn(4) on an element with source point can be mapped to a parametric plane, the integration is converted to:

$$I^e = \iint_{S^e-e^\epsilon} f(x)\frac{\partial^2 G(\mathbf{x},\mathbf{y})}{\partial y_3 \partial n} ds = \sum^a f^a \iint_{S^e-e^\epsilon} N^a(\xi)\frac{\partial^2 G(\mathbf{x}(\xi),\mathbf{y})}{\partial y_3 \partial n}|J(\xi)|d\xi_1 d\xi_2 \tag{5}$$

$$= \sum^a f^a \int_{\theta_1}^{\theta_2}\int_{\alpha(\epsilon,\theta)}^{\hat{\rho}(\theta)} F(\rho,\theta)d\rho d\theta \tag{6}$$

Guiggani (1992) expanded each component of $F(\rho,\theta)$ in Eqn(6) around the source point using Lawrence expansion to $O(\rho^{-2})$ like Eqn (7). With some extra mathematical work, the three integrations in Eqn (7) can be evaluated using common methods.

$$F(\rho,\theta) = F_{-2}\cdot\rho^{-2} + F_{-1}\cdot\rho^{-1} + F_0 + O(\rho^{-3}) \tag{7}$$

Gao (2010) developed a more general method. Similar to Eqn(5), the hypersingular integral on an element is given by Eqn(8) and can be turned into a line integral along the edges of the element like Eqn(9).

$$I^e_a = \int_{S^e} \frac{\bar{f}(\mathbf{x},\mathbf{y})}{r^\lambda(\mathbf{x},\mathbf{y})} ds = \int_{-1}^{1}\int_{-1}^{1} \frac{\bar{f}(\mathbf{x}(\xi),\mathbf{y})}{r^\lambda(\mathbf{x}(\xi),\mathbf{y})}|J(\xi)|d\xi_1 d\xi_2 \tag{8}$$

$$= \int_L \frac{1}{\rho(\mathbf{x},\mathbf{y})}\frac{\partial \rho(\mathbf{x},\mathbf{y})}{\partial n'} F(\mathbf{x},\mathbf{y})dL \tag{9}$$

[1]Unbounded term and some limit notations are neglected for simplicity.

where

$$F(\mathbf{x},\mathbf{y}) = \lim_{\rho\alpha(\epsilon)\to 0} \int_{\rho\alpha(\epsilon)}^{\rho(\mathbf{x},\mathbf{y})} \frac{\bar{f}(\hat{x},\mathbf{y})}{r^{\lambda}(\hat{x},\mathbf{y})} J\rho d\rho \tag{10}$$

and $r = ||\mathbf{x}-\mathbf{y}||_2$, $\rho(\mathbf{x},\mathbf{y}) = ||\mathbf{x}-\mathbf{y}||_2$, n' is the normal vector on the square in parametric plane pointing outward. Evaluation of Eqs (9) and (10) will not be elaborated in this abstract, but the essential idea is to convert the non-singular part of integrand to a power series which can be used later for eliminating the singularity.

4. Evaluation of Free Term Coefficients

For an inner domain problem, the coefficients in Eqn(4) can be evaluated using an indirect method. By assuming $\phi(\mathbf{y}) = 1$, $\phi(\mathbf{y})_x = 0$, $\phi(\mathbf{y})_y = 0$ and $\phi(\mathbf{y})_z = 0$, we can obtain:

$$a_3 = \iint_S \frac{\partial^2 G(\mathbf{x},\mathbf{y})}{\partial y_3 \partial n} ds(x) \tag{11}$$

Again, by assuming $\phi(\mathbf{y}) = x$, $\phi(\mathbf{y})_x = 1$, $\phi(\mathbf{y})_y = 0$ and $\phi(\mathbf{y})_z = 0$, we can obtain:

$$C_{31} = \iint_S [\frac{\partial^2 G(\mathbf{x},\mathbf{y})}{\partial y_3 \partial n} x - \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial y_3} n_1] ds - a_3 y_1 \tag{12}$$

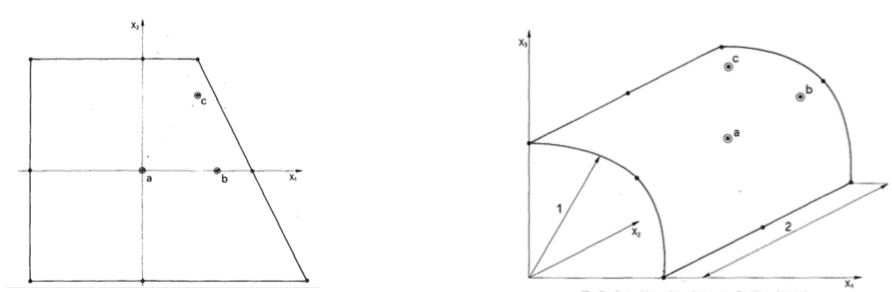
Similarly, we can evaluate C_{32} and C_{33}. For an outer domain problem, the indirect method is not applicable due to the integration cannot be carried out over a complete domain conveniently. However, by looking at the analytical expression of these coefficients, the following relationships can be established between the coefficients of the inner and the outer domains at the waterline:

$$\begin{aligned} a_3^i + a_3^o &= 0 \qquad & C_{31}^i + C_{31}^o &= 0 \\ C_{32}^i + C_{32}^o &= 0 \qquad & C_{33}^i + C_{33}^o &= 0.5 \end{aligned} \tag{13}$$

For other free surface nodes, $C_{31} = 0$, $C_{32} = 0$, $C_{33} = 0.5$ and $a_3 = 0$. Hence, the coefficients for outer domain problem can be evaluated by calculating the coefficients for the inner domain.

5. Numerical Tests

The evaluation program of hypersingular integral is tested for two different cases. The results are compared with Guiggani's results(1992) and analytical results. The validation results are listed in Table. 1.



(a) Flat Element (b) Curved Element

Figure 1: Hypersingular integral validation cases

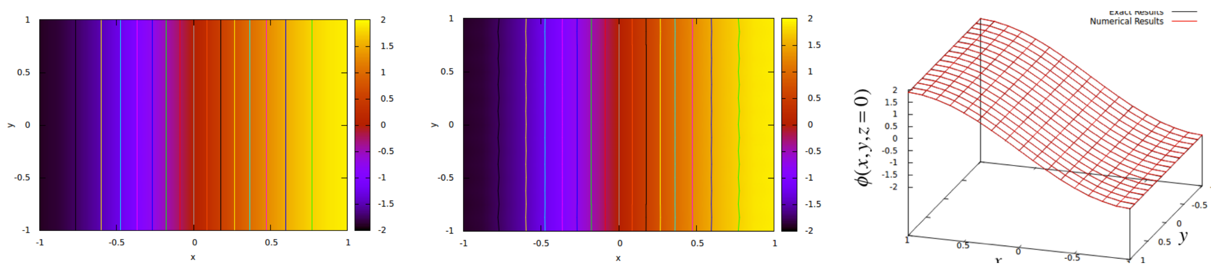
Table 1: Validation of hypersingular integrals

Flat Element			
Point	Our code	Guiggani	Analytical
a	-5.74515	-5.74924	-5.74937
b	-9.15700	-9.15744	-9.15459
c	-15.32742	-15.30541	-15.32850

Curved Element			
Point	Our code	Guiggani	Analytical
a	-0.343923	-0.343804	-0.343807
b	-0.497122	-0.497091	-0.497099
c	-0.876357	-0.877106	-0.877214

The new BIE is tested for a box domain. ϕ is given on the top face and $\partial\phi/\partial n$ is given on the other faces as boundary condition. The potential over the domain is set to be Eqn (14). Results are shown in Figure 2.

$$\phi(\mathbf{x}) = \frac{cosh\ k(z+d)}{cosh(kd)} e^{ikx} \tag{14}$$



(a) Analytical Result

(b) Numeral Result

(c) Potential at Surface

Figure 2: Comparison of results for an inner domain case

6. Summary

In this paper, we established new boundary integral equations for BEM to get diagonal dominant coefficient matrix. A numerical test of inner domain problem was carried out to validate if the new method is working properly. Further extension to time domain problem will be presented at the workshop.

References

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