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**Instability of Axially-Symmetric Propagating Waves by a Vertically-Oscillating Sphere**

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# 1 Abstract

We study the instability of axially-symmetric propagating waves that are generated by a vertically-oscillating sphere in a circular basin. Laboratory experiments have demonstrated that when the amplitude of body oscillation exceeds a threshold value, the radiated axially-symmetric waves abruptly transfigure into non-axially symmetric waves. The purpose of this work is to understand the fundamental mechanism governing the occurrence of this phenomenon. We employ the transition matrix method to analyze the instability of the nonlinear time periodic wave-body interaction system by use of fully-nonlinear (time-domain) numerical simulations for describing temporal and spatial evolution of the base and disturbed propagating wave fields. We quantify the initial growth rate and dominant modes of unstable disturbances and study their dependence on physical parameters including body-oscillation frequency and amplitude, body geometry, surface tension and basin size. The theoretical prediction compares well with the existing experimental observations for unstable mode shape and growth rates. We show that the instability is associated with the subharmonic resonant interactions of axisymmetric base flow and non-axisymmetric disturbances. In addition, we investigate the long-time evolution dynamics of the unstable wave-body system including characterization of wave patterns and hydrodynamic loads on the body.

# 2 Introduction

When a half-submerged sphere is forced to oscillate vertically, axially-symmetric outgoing ring waves are generally expected to be produced. However, laboratory experiments show that the distinctive pattern of the generated waves varies with the amplitude of sphere oscillation (Tatsuno, Inoue & Okabe 1969; Taneda 1991). Specifically, when the amplitude of body oscillation (at a given oscillation frequency) increases beyond a threshold value, the axially-symmetric ring wave pattern disappears while non-axially symmetric wave patterns are developed abruptly. Despite its fundamental importance, there has been a limited theoretical and numerical studies to understand the mechanism that causes this phenomenon.

Becker & Miles (1991) used an averaged Lagrangian method to study the instability of standing waves in an annular by considering the inner cylinder as a wave maker. They found that when the cross-wave (disturbance) frequency is close to one half of the frequency of the forced standing ring wave, the cross-wave becomes unstable, which can cause the wave pattern to be changed from axisymmetric to non-axisymmetric shape. But this study cannot be applied to explain the above stated experimental observation since the base wave is progressive but not standing wave. To overcome this limitation, Becker & Miles (1992) carried out a modulational instability analysis for propagating waves around a vertical cylinder. The theoretical prediction, however, does not correlate with the experimental measurements of Tatsuno, Inoue & Okabe (1969). It remains unclear what mechanism causes the experimentally-observed phenomenon.

In this work, we numerically investigate the instability of axially-symmetric outgoing waves in a circular basin based on the transition matrix method. We use fully-nonlinear numerical simulations to obtain the description of the nonlinear base and disturbed wave flows for building up the transition matrix. The predicted characteristic unstable wave patterns and growth rates are verified by quantitative comparisons with the available experimental data. The mechanism governing the occurrence of the observed phenomenon is then deduced.

# 3 Methodology

## 3.1 Nonlinear Wave Generation by an Oscillating Body

Figure 1 illustrates the physical problem of wave generation and propagation by a vertically-oscillating sphere in a cylindrical basin. The sphere has a radius of $R$ and is half submerged (at the mean position). The origin of the sphere is located at the vertical axis of the cylindrical basin that has a radius of $R_2$ and a depth of $H$. The potential flow for the wave motion is assumed. The velocity potential $\phi(x, y, z, t)$ satisfied the Laplace equation anywhere in the fluid and the nonlinear free-surface boundary conditions:

$$\phi_t + \frac{1}{2}\nabla\phi \cdot \nabla\phi + g\zeta + \frac{\sigma}{\rho}\nabla \cdot \mathbf{n} = 0, \qquad z = \zeta \tag{1}$$

$$\zeta_t + \zeta_x\phi_x + \zeta_y\phi_y = \phi_z, \qquad z = \zeta \tag{2}$$

where $\zeta(x, y, t)$ is the free-surface elevation, $\mathbf{n}$ the unit normal pointing out of the fluid, $\sigma$ the surface tension coefficient, $\rho$ the water density, and $g$ the gravitational acceleration. The boundary condition on the instantaneous sphere surface is

$$\phi_n = \dot{\xi} n_z \tag{3}$$

where $\xi(t) = A \sin \omega t$ represents the forced periodic vertical motion of the sphere (with frequency $\omega$ and amplitude $A$). The zero normal flux ($\phi_n$=0) condition is imposed on the wall and bottom of the basin. To avoid wave reflection by the basin wall, a narrow strip of sponge layer is placed along the tank, as in the experiments (Tatsuno, Inoue & Okabe 1969).

A fully-nonlinear quadratic boundary element method (QBEM) (e.g. Yan 2010) is employed to solve the stated nonlinear initial-boundary-value problem (with any specified initial conditions) in the time domain. With QBEM simulations, we obtain the time evolution of the generated nonlinear wave field in the basin.

### 3.2 Instability Analysis by the Transition Matrix Method

Since the waves generated by sphere oscillation propagate outwards in the radial direction, the base flow is time periodic and cannot be made steady through a coordinate-system transform (such as Stokes waves). The conventional instability-analysis approaches for steady flows (e.g. McLean 1982) cannot be applied here. In the present case, the so-called transition matrix method needs to be used. In this approach, we write the governing equation for time evolution of the nonlinear wave-body system in a symbolic form:

$$\frac{\partial \mathbf{u}}{\partial t} = \mathcal{N}(\mathbf{u}), \tag{4}$$

where $\mathbf{u} = (\phi(\mathbf{x}, t), \zeta(\mathbf{x}, t))$, $\mathbf{x} \equiv (x, y)$, $\mathcal{N}$ is a nonlinear operator (cf. equations (1) and (2)). In a linear stability analysis, we write $\mathbf{u}$ as the sum of base flow (corresponding to the propagating axially-symmetric wave) $\mathbf{u_0}$, and a small disturbance $\mathbf{u}'$:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u_0}(\mathbf{x}, t) + \mathbf{u}'(\mathbf{x}, t) \ . \tag{5}$$

Substituting (5) into (4) leads to the linearized governing equation for disturbance evolution:

$$\frac{\partial \mathbf{u}'}{\partial t} = \mathcal{L}(\mathbf{u}') \tag{6}$$

where $\mathcal{L}$ is the linearized time-periodic operator. For a disturbance with $N$ degrees of freedom, we express the solution of (6) as a combination of $N$ linearly independent solutions $\mathbf{u_i'}$, $i = 1, \ldots, N$:

$$\mathbf{u}' = \sum_{i=1}^{N} \gamma_i \mathbf{u}_i' \tag{7}$$

where the coefficients $\gamma_i$, $i = 1, \ldots, N$, are obtained from the initial condition for $\mathbf{u}'$. We further expand each $\mathbf{u}_i'$ in terms of normal modes:

$$\mathbf{u'}_i(\mathbf{x}, t) = \sum_{j=1}^{N} \mu_{ij}(t)\Psi_j(\mathbf{x}) \tag{8}$$

where $\Psi_j(\mathbf{x})$, $j = 1, \ldots, N$, are spatial basis functions given by the homogeneous solution of the linearized wave-body boundary-value problem, and $\mu_{ij}(t)$, $i, j = 1, \ldots, N$, the time-dependent modal amplitudes. Substituting (8) and (7) into (6), we obtain from Floquet theory (Coddington & Levinson 1955) the solution of the modal amplitudes:

$$\mathcal{U}(t) = \mathcal{P}(t) \exp(\mathcal{B}t) \tag{9}$$

where $\mathcal{U} = [\mu_{ij}]$ is an $N \times N$ matrix, $\mathcal{P}(t)$ is a time-periodic $N \times N$ matrix and $\mathcal{B}$ a time-independent $N \times N$ matrix. The stability of the system is determined by the eigenvalue $\lambda_i$ $(i = 1, \ldots, N)$ of $\mathcal{B}$. The flow is unstable if any $\Re(\lambda_i) > 0$ with the unstable mode being the eigenvector corresponding to $\lambda_i$.

Because $\mathcal{L}$ is not explicitly known, it is difficult to derive $\mathcal{B}$ analytically. We generally resort to a numerical determination from (6) of the transition matrix $\mathcal{Q} \equiv \exp(\mathcal{B}T)$ where $T$ is the wave period. In terms of eigenvalues $\sigma_i$ of $\mathcal{Q}$, the eigenvalues of $\mathcal{B}$ is:

$$\lambda_i = \frac{\log \sigma_i}{T}, \qquad i = 1, \ldots, N. \tag{10}$$

For a given base flow $\mathbf{u}_0$, we obtain the transition matrix $\mathcal{Q}$ by evolving the nonlinear wave-body system in time for one period to get $\mathbf{u}(T)$ with the perturbed initial condition $\mathbf{u}_0(0) + \delta\Psi_i(\mathbf{x})$, $i = 1, \ldots, N$, $\delta \ll 1$ (e.g. Zhu, Liu & Yue 2003). This is achieved by use of fully-nonlinear QBEM simulations.

# 4 Results

Figure 2 shows the wave patterns resulted from the forced vertical oscillation of the sphere with small ($A$=1 mm) and large ($A$=2.5 mm) motion amplitudes. The results are obtained from direct numerical simulations by QBEM. Other physical parameters are $kR$=16.1 $R_2/R$=5.5, $H/R$=4, and surface tension effect is included. The numerical simulations clearly show that the wave pattern loses its axial symmetry when the sphere oscillation amplitude increases from $A$=1 mm ($kA$=0.54) to $A$=2.5 mm ($kA$=1.35). The characteristic features of the wave patterns predicted by numerical simulations agree well with the experimental observation of Tatsuno, Inoue & Okabe (1969) and Taneda (1991). Figure 3 compares the time history of the wave elevation at a fixed point in the wave field, generated by sphere motion of $kA$=1.35, with and without the presence of a small initial disturbance. The result indicates that the sub-harmonic wave component is unstable and experiences significant growth causing the formation of non-axially symmetric wave pattern. Systematic results of the instability analysis based on the transition matrix method will be presented in the workshop.

# 5 Reference


Becker, J.M. & Miles, J.W. 1991 Standing radial cross-waves. *J. Fluid Mech.* **222**: 471-499.

Becker, J.M. & Miles, J.W. 1992 Progressive radial cross-waves. *J. Fluid Mech.* **245**: 29-46.

Coddington, D.A. & Levinson, N. 1955 *Theory of Ordinary Differential Equations.* McGraw-Hill, New York.

McLean, J.W. 1982 Instabilities of finite-amplitude water waves. *J. Fluid Mech.* **114**:315-330.

Taneda, S. 1991 Visual observations of the flow around a half-submerged oscillating sphere. *J. Fluid Mech.* **227**:193-209.

Tatsuno, M., Inoue, S. & Okabe, J. 1969 Transfiguration of surface waves. *Rep. Res. Inst. Appl. Mech. Kyushu University* **17**: 195-215.

Yan, H. 2010 Computations of fully nonlinear 3D wave-body Interactions. PhD Thesis, Massachusetts Institute of Technology, USA

Zhu, Q., Liu, Y. & Yue, D. K. P. 2003 Three-dimensional instability of standing waves. *J. Fluid Mech.* **496**: 213-246.

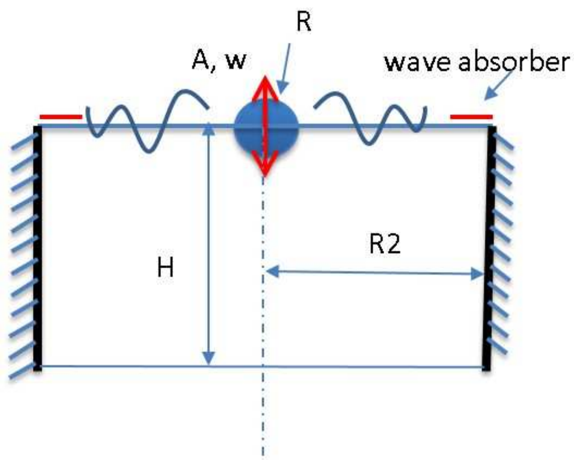


Figure 1: Sketch of the physical problem.

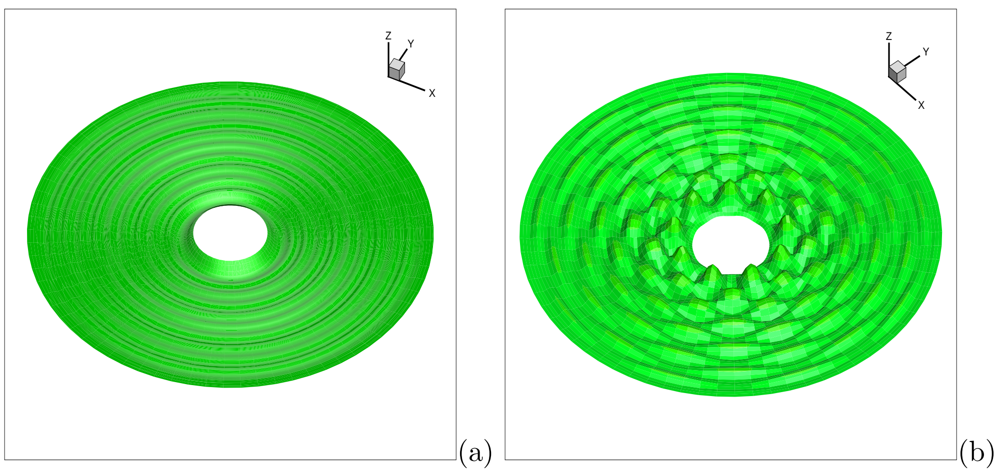


Figure 2: Free surface pattern of the wave field resulted from vertical sphere motion with: (a) small and (b) large oscillation amplitudes.

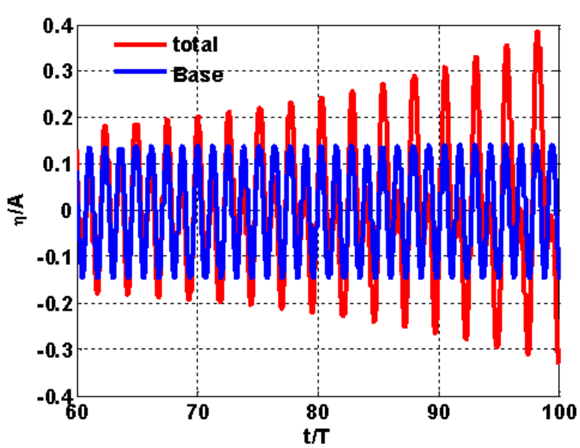


Figure 3: Time histories of wave elevation of a point in the base wave field (blue line) and disturbed wave field (red line). The difference between two curves represents the growth of the disturbance.