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Nonlinear Loads on a Vertical Circular in Irregular Waves

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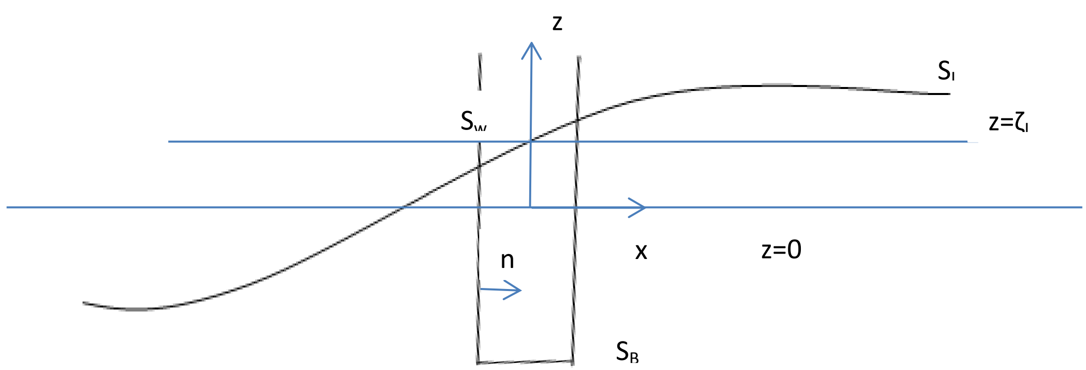
Summary

A new expression is derived for the nonlinear surge force acting on a vertical circular cylinder in irregular waves. A recently developed fluid impulse theory for the nonlinear loads on floating bodies is invoked to express the total force as the time derivative of a Froude-Krylov, diffraction and free surface impulse. The impulses involve integrals of the incident and diffraction velocity potentials over the body boundary and an internal waterplane area which coincides with the instantaneous position of the ambient wave profile. The evaluation of the partial time derivative and gradient of the disturbance potential over the body boundary is circumvented. An expression for the nonlinear force for moderate wavelengths leads to an efficient computational method. For wavelengths large compared to the cylinder diameter the nonlinear force reduces to GI Taylor's formula with a quadratic and a cubic point load at the waterline.

Fluid Impulse Theory

The Figure illustrates a vertical circular cylinder fixed in space, an ambient wave profile and the calm water level coinciding with the z=0 plane. The positive z-axis points upwards and the unit normal vector points inside the cylinder. The wetted surface under the ambient wave profile is $S_B(t)$ the instantaneous ambient wave surface exterior to the cylinder is $S_I(t)$ and its portion inside the cylinder is $S_W(t)$. The water density is ρ and the acceleration of gravity is g.



The fluid impulse theory developed by Sclavounos (2012) expresses the nonlinear surge force on the cylinder as the sum of the time derivative of Froude-Krylov, diffraction and free-surface impulse components

$$F_X(t) = F_{X,F-K}(t) + F_{X,B} + F_{X,FS}$$

Denote by φ_I and φ the incident and diffraction velocity potentials, respectively. The Impulse Froude-Krylov force is given by

$$F_{X,F-K}(t) = -\rho \frac{d}{dt}\left[\int_{S_B(t)+S_W(t)} \varphi_I \, n_1 \, ds \right] = \rho \frac{d}{dt}\left[\iint_{\forall_W(t)} \frac{\partial \varphi_I}{\partial x} dv \right] \tag{1}$$

where Gauss' theorem was invoked. The Impulse Diffraction force is given by

$$F_{X,B} = -\rho \frac{d}{dt}\left[\int_{S_B(t)} \varphi n_1 \, ds \right] \tag{2}$$

The free surface Impulse Force in the x-direction takes the form

$$F_{X,FS} = -\rho \frac{d}{dt} \int_{S_I(t)} \varphi n_1 \, ds - \rho \frac{d}{dt} \int_{S_I(t)} \frac{\partial \zeta}{\partial x} \frac{\partial \varphi_I}{\partial x} ds - \rho \frac{d}{dt} \int_{S_I(t)} \zeta \frac{\partial \varphi}{\partial x} ds + O(\varepsilon^3) \tag{3}$$

where ζ is the diffraction wave elevation about the ambient wave profile. Terms of cubic order in the wave steepness ε are neglected in (3). A surface integral over a semi-spherical surface at infinity involving the inner product of the gradient of the incident and diffraction potentials may be shown to vanish. At a finite time the diffraction potential at infinity behaves like a 3D dipole and its gradient decays like $1/R^3$. This decay offsets the area of the sphere which grows like R^2.

Invoking the linear free surface condition we obtain to leading order in the wave steepness:

$$n_1 = -\frac{\partial \zeta_I}{\partial x} = \frac{1}{g} \frac{\partial^2 \varphi_I}{\partial x \partial t}, \; z = \zeta_I; \qquad \frac{\partial \zeta}{\partial x} = -\frac{1}{g} \frac{\partial^2 \varphi}{\partial x \partial t}, \; z = \zeta_I \tag{4}$$

Introducing (4) in (3), evaluating the integrals over the $z=\zeta_I$ plane with errors of $O(\varepsilon^3)$ and interchanging the time derivative with the surface integral, it follows after some simple algebra that

$$F_{X,FS} = -\frac{\rho}{g} \int_{z=\zeta_I} \left[\varphi \frac{\partial}{\partial x}\left(\frac{\partial^2 \varphi_I}{\partial t^2} \right) - \frac{\partial^2 \varphi}{\partial t^2} \frac{\partial \varphi_I}{\partial x} \right] ds + \frac{\rho}{g} \frac{d}{dt} \int_{z=\zeta_I} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial x} ds + O(\varepsilon^3) \tag{5}$$

Invoking the linearized free surface condition of the incident and diffraction potentials in the first term of equation (5) we obtain

$$F_{X,FS} = \rho \int_{z=\zeta_I} \left[\varphi \frac{\partial}{\partial z}\left(\frac{\partial \varphi_I}{\partial x} \right) - \frac{\partial \varphi}{\partial z} \frac{\partial \varphi_I}{\partial x} \right] ds + \frac{\rho}{g} \frac{d}{dt} \int_{z=\zeta_I} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial x} ds + O(\varepsilon^3) \tag{6}$$

Since the z-derivative is equal to the normal derivative on the $z=\zeta_I$ plane, the first term in the right-hand side of (6) may be reduced to an integral over the body boundary by invoking Green's identity for the potentials $\varphi_1 = \varphi, \; \varphi_2 = \partial \varphi_I / \partial x$ over a closed surface consisting of the $z=\zeta_I$ plane, the body surface and a sphere at infinity with radius R. Over the sphere the dipole-like diffraction potential decays like $1/R^2$

offsetting the R^2 growth of the surface of the sphere. The remaining integral involves the oscillatory incident wave potential which as $R \to \infty$ decays like $R^{-1/2}$.

Following the application of Green's theorem and invoking the body boundary condition $\partial\varphi / \partial n = -\partial\varphi_I / \partial n$ on S_B, it follows that

$$F_{X,FS} = -\rho \int_{S_B} \left[\varphi \frac{\partial}{\partial n}\left(\frac{\partial\varphi_I}{\partial x}\right) + \frac{\partial\varphi_I}{\partial n}\frac{\partial\varphi_I}{\partial x} \right] ds + \frac{\rho}{g}\frac{d}{dt}\int_{z=\zeta_I} \frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x} ds = F^B{}_{X,FS} + F^{FS}{}_{X,FS} \qquad (7)$$

The force expressions (1) and (2) involve no linearization assumptions. In the force expression (7) terms of cubic order in the wave steepness were omitted. The ambient waves are assumed to be irregular and expressions (1), (2) and (7) are valid for finite values of Ka, where K is a characteristic wavenumber and a is the cylinder radius.

The characteristic wavelength in a seastate is often large relative to the cylinder diameters of offshore structures and wind turbines. In such cases Ka is a small parameter and the diffraction potential near the cylinder may be approximated to leading order by the 2D cross-flow potential

$$\varphi(r,\theta) = -u_1 \frac{a^2}{r}\cos\theta, \qquad u_1 = \frac{\partial\varphi_I}{\partial x}(r=0) \qquad (8)$$

Introducing (8) in (7) and expressing $\partial\varphi / \partial x$ in polar coordinates, the second term in the right hand side, $F^{FS}{}_{X,FS}$, which involves a quadratic product of the diffraction potential vanishes identically. The first term in the right-hand side of (7) is the leading-order free-surface impulse force.

Substituting (8) in (1) & (2), the Impulse Froude-Krylov force (1) and Impulse diffraction force (2) become equal in the limit of small Ka and their sum is

$$F_{X,F-K} + F_{X,B} = 2\rho\pi a^2 \int_{-T}^{\zeta_I} dz\, \dot{u}_1(z)\Big|_{x=0} + 2\rho\pi a^2 \frac{\partial\zeta_I}{\partial t} u_1\Big|_{x=0,z=\zeta_I} \qquad (9)$$

The first term in the right-hand side of (9) is the inertia component of Morison's equation. The second term is a point quadratic load acting on the waterline.

Substituting (8) in (7), recalling that $n_1 = -\cos\theta$ on the cylinder boundary and denoting by $\vec{v} = (u_1, u_2, u_3) = \nabla\varphi_I$ we obtain

$$F^B{}_{X,FS} = \rho \int_{S_B} \left[a\, u_1\, n_1 \frac{\partial u_1}{\partial n} - u_1(\vec{v}\cdot\vec{n}) \right] ds = \rho \int_{S_B} \left[a\, u_1\, n_1 \left(\vec{n}\cdot\nabla u_1\right) - u_1(\vec{v}\cdot\vec{n}) \right] ds \qquad (10)$$

For a unidirectional wave $u_2 = 0$. On the vertical boundary of the cylinder $n_3 = 0$, therefore $\vec{n} \cdot \nabla u_1 = n_1 u_{1x}$. On the cylinder bottom $n_1 = 0$, therefore the first term of the integrand in (10) becomes $a n_1^2 u_1 u_{1x}$. Invoking Gauss' theorem in the second term of the integrand we obtain

$$\int_{S_B} u_1(\vec{v} \cdot \vec{n})\, ds = \int_{S_B} u_1(u_j n_j)\, ds = -\int_{\forall} \frac{\partial}{\partial x_j}\left(u_1 u_j\right) dv + \int_{S_w} u_1 u_3 ds = -\int_{\forall} u_j \frac{\partial u_1}{\partial x_j} dv + \int_{S_w} u_1 u_3 ds \qquad (11)$$

In (11) an integral over the waterplane area S_W on $z=\zeta_I$ was added and subtracted in order to apply Gauss' theorem over a closed surface and the mass conservation principle was invoked. Using the nonlinear kinematic free surface condition on $z=\zeta_I$ and combining expressions (10) and (11) we obtain

$$F^B{}_{X,FS} = 2\rho\pi a^2 \int_{-T}^{\zeta_I} (u_1 u_{1x} + u_3 u_{1z})\, dz - \rho\pi a^2 u_1\, u_3\big|_{x=0,z=\zeta_I}\,, \quad u_3 = \frac{\partial \zeta_I}{\partial t} + u_1 \frac{\partial \zeta_I}{\partial x} \qquad (12)$$

The expression (12) added to (9) leads to GI Taylor's force expression which accounts for the convective terms of the ambient wave acceleration [Newman (1977)]. Expression (12) also includes a point load at the waterline. The small Ka expression for the total force is the sum of (9) and (12)

$$F^B{}_{X,FS} = 2\rho\pi a^2 \int_{-T}^{\zeta_I} \left(\dot{u}_1 + u_1 \frac{\partial u_1}{\partial x} + u_3 \frac{\partial u_1}{\partial z} \right) dz + \rho\pi a^2 u_1 \left(\frac{\partial \zeta_I}{\partial t} - u_1 \frac{\partial \zeta_I}{\partial x} \right)_{x=0,z=\zeta_I} \qquad (13)$$

The higher order correction to the diffraction potential (8) was shown by Faltinsen, Newman and Vinje (1995) to be of $O(Ka)^3$. This is of the same order as that of the second term in the right-hand side of expression (7) which is neglected. Second-order diffraction effects are therefore dominated by quadratic products of the incident wave disturbance and its cross products with the diffraction disturbance. The nonlinear force (1), (2) & (7) hence neglects effects which are quadratic in the diffraction wave disturbance. Its small Ka approximation (13) reduces to the GI Taylor formula which accounts for up to cubic effects above the z=0 plane and a point load at the waterline which includes quadratic and a cubic effects, all expressed as explicit functions of the ambient wave kinematics.

Acknowledgements

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