A 2-D and 3-D Generalized Wagner Method Using a High-order Boundary Element Method and Acceleration Potential

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Introduction

The impact of marine vessels in water has been the subject of considerable study. The large forces that occur during impact have consequences for structural design and can influence the motions of the vessel. Therefore the prediction of impact loads is an important aspect of ship design.

The starting point for this work comes from Zhao et al. [4]. In that paper, the Generalized Wagner Method is introduced, which accounts for the exact body boundary condition, while maintaining a horizontal free surface on which the high-speed free-surface boundary condition is applied. The position of the free surface is updated to account for the water run-up along the body.

In this abstract, an extension of the Generalized Wagner Method described in Zhao et al. [4] is made for a high-order boundary element method. The boundary element method uses B-splines to represent the flow quantities, allowing for smooth variations with relatively few elements. An additional extension is made by using an acceleration potential to obtain accurate forces, which requires a special treatment of the waterline expansion rate. The treatment of the waterline expansion term is different than previous work, such as Bandyk and Beck [1]. After the validation of the high-order two-dimensional method, it is exented into three dimensions.

Methodology

High order BEM Green's Identity in three-dimensions is given in Equation 1. The 3D free-space Green's function $G = 1/r$ is used, where $r = |\vec{x} - \vec{\xi}|$.

$$-2\pi\phi\left(\vec{x},t\right) = \iint_{S(t)} \left[G_n\left(\vec{x},\vec{\xi}\right)\phi\left(\vec{\xi},t\right) - G\left(\vec{x},\vec{\xi}\right)\phi_n\left(\vec{\xi},t\right)\right] dS\left(\vec{\xi}\right) \tag{1}$$

Green's Identity in two-dimensions is given in Equation 2. The 2D free-space Green's function $G = \ln r$ is used.

$$\pi\phi\left(\vec{x},t\right) = \int_{a(t)}^{b(t)} \left[G_n\left(\vec{x},\xi\right)\phi\left(\xi,t\right) - G\left(\vec{x},\xi\right)\phi_n\left(\xi,t\right)\right] dS(\xi) \tag{2}$$

The potential ϕ and its normal derivative ϕ_n are represented on the boundaries with B-splines. For this work quadratic B-splines are used, and thus spatial derivatives up to second order along the boundary surfaces can be computed analytically.

Boundary Conditions The boundary condition on the free surface is $\phi = 0$, which is the high-speed limit, and is the common condition for impact problems. Similarly, $\phi_t = 0$ on the free surface. The body boundary condition is $\phi_n = \dot{\vec{\delta}}$ and $\phi_{nt} = \ddot{\vec{\delta}}$, where $\vec{\delta}$ is the body displacement vector.

Acceleration potential Due to the large velocities in impact problems, the time derivatives in the problem will also be large. The pressure, given by Equation 3, depends on the time derivative of the potential, and thus it is important to accurately calculate that derivative to obtain accurate forces.

$$p = p_\infty - \rho\frac{\partial\phi}{\partial t} - \frac{1}{2}\rho\left|\nabla\phi\right|^2 - \rho g z \tag{3}$$

Therefore, anaylical derivatives are prefered over numerical derivatives. Note that for impact problems, the hydrostatic term $\rho g z$ is often neglected as much smaller than the other terms. By solving directly for the time derivative of the potential, pressure can be obtained without numerical derivatives. Taking the partial derivate of Green's identity with respect to time and using Leibniz's Rule for the time dependent wetted surface yeilds:

$$
\begin{aligned}
-2\pi\phi_t(\vec{x},t) = &\iint_{S(t)} \left[G_n\left(\vec{x},\vec{\xi}\right)\phi_t\left(\vec{\xi},t\right) - G\left(\vec{x},\vec{\xi}\right)\phi_{nt}\left(\vec{\xi},t\right)\right] dS\left(\vec{\xi}\right) \\
&+ \iint_{S(t)} \nabla\cdot\left[\phi\left(\vec{\xi},t\right)\nabla G\left(\vec{x},\vec{\xi}\right) - G\left(\vec{x},\vec{\xi}\right)\nabla\phi\left(\vec{\xi},t\right)\right](\vec{v}\cdot\hat{n})\, dS\left(\vec{\xi}\right) \\
&- \oint_{\partial S(t)} \left(\vec{v}\times\left[\phi\left(\vec{\xi},t\right)\nabla G\left(\vec{x},\vec{\xi}\right) - G\left(\vec{x},\vec{\xi}\right)\nabla\phi\left(\vec{\xi},t\right)\right]\right)\cdot d\vec{s} \qquad (4)
\end{aligned}
$$

Here $\vec{v}$ is the velocity of the surface and the second term cancels out because G and ϕ satisfy Laplace's equation. In two-dimensions the equation for ϕ_t becomes:

$$
\begin{aligned}
\pi\phi_t(\vec{x},t) = &\int_{a(t)}^{b(t)} \left[G_n(\vec{x},\xi)\phi_t(\xi,t) - G(\vec{x},\xi)\phi_{nt}(\xi,t)\right] dS(\xi) \\
&+ \left[G_n(\vec{x},b(t))\phi(b(t),t) - G(\vec{x},b(t))\phi_n(b(t),t)\right]\frac{db}{dt} \\
&- \left[G_n(\vec{x},a(t))\phi(a(t),t) - G(\vec{x},a(t))\phi_n(a(t),t)\right]\frac{da}{dt}
\end{aligned} \qquad (5)
$$

The terms da/dt and db/dt become the waterline expansion rates for impact problems. From this equation, it becomes clear that ϕ, ϕ_n, G, and G_n will be evaluated at the ends of the patches, and that the singularity of the Green's function must be removed. To accomplish this, the acceleration potential problem is solved using the negative image method. Using the image method, the sources will cancel at the free surface, and the dipoles will become a single vertical dipole. Analytically, the flow around end of the body is a corner flow, with $\phi = Cz^{\alpha}$, and the power α is related to the local deadrise angle β by Equation 6.

$$\alpha = \frac{1}{2 - 2\beta/\pi} \qquad (6)$$

The edges of the body surface are then at $(c(t), \eta(t))$ and $(-c(t), \eta(t))$, and $da/dt = dc/dt$ and $db/dt = -dc/dt$. The calculation of the wetted surface expansion rate dc/dt is treated in the next section.

Calculation of expansion rate The goal of using the acceleration potential is to eliminate numerical derivatives. Therefore, it is desirable to have an analytical expression for the wetted area expansion rate dc/dt. This is achieved using a Wagner-like method, where the relative vertical velocity on the free surface is integrated to find the intersection with the body surface $\eta_b(y)$. The method for the classic Wagner problem (flat plate approximation) is shown in Equation 7 [2]:

$$\eta_b(y) = \int_0^t \frac{V|y|}{\sqrt{y^2 - c^2(\tau)}} d\tau = \int_0^y \frac{y\mu(c)}{\sqrt{y^2 - c^2}} dc \qquad (7)$$

Here, $\mu(c)dc = V dt$, or $dc/dt = V/\mu(c)$, where V is the impact velocity. A form for $\mu(c)$ must be assumed, and here it is assumed that $\mu(c) = A$, a constant. By using $\eta_b(y) = y\tan\beta$ for a wedge, the well known Wagner result is obtained.

$$\frac{dc}{dt} = \frac{V}{\mu(c)} = \frac{V}{A} = \frac{\pi}{2}\frac{V}{\tan\beta} \qquad (8)$$

Using the power law for corner flow for the velocity at $y = c$ and the asmptotic behavior that the vertical velocity goes to V as $y \to \infty$, a similar analysis yeilds:

$$\eta_b(y) = \int_0^y \frac{Ay^{2(1-\alpha)}}{\left(y^2 - c^2\right)^{1-\alpha}} dc = A\int_0^y \left(\frac{y^2}{y^2 - c^2}\right)^{1-\alpha} dc = A\int_0^y \left(\frac{1}{1 - c^2/y^2}\right)^{1-\alpha} dc \qquad (9)$$

Defining $u = (c/y)^2$, then $du = 2(c/y)(dc/y) = (2/y)u^{1/2}dc$ and $dc = (y/2)u^{-1/2}du$. Substituting in yeilds:

$$\eta_b(y) = A\int_0^1 (1-u)^{\alpha-1}\left(\frac{y}{2}u^{-\frac{1}{2}}du\right) = A\frac{y}{2}\int_0^1 u^{-\frac{1}{2}}(1-u)^{\alpha-1}du = A\frac{y}{2}B\left(\frac{1}{2},\alpha\right) \qquad (10)$$

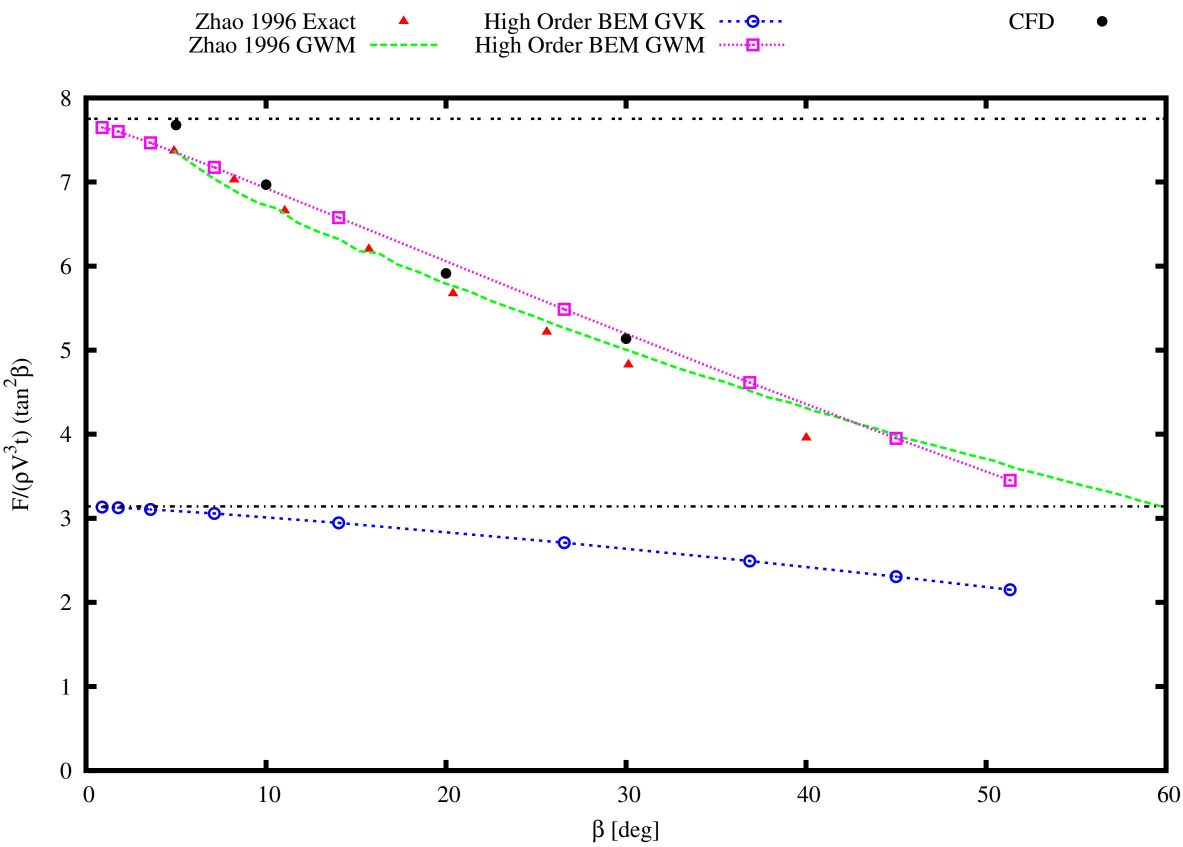

Figure 1: Non-dimensional force comparison between results of [4] and present method. The horizontal lines mark classical von Karman (lower) and Wagner (upper) methods.

Here, $B\left(\frac{1}{2},\alpha\right)$ is the beta function evaluated at $1/2$ and α. Reformulated in terms of gamma functions:

$$\eta_b(y) = A\frac{y}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{1}{2}+\alpha\right)} = A\frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha)}{\Gamma\left(\frac{1}{2}+\alpha\right)}y = y\tan\beta \tag{11}$$

Thus, the expression for the waterline expansion rate is:

$$\frac{dc}{dt} = \frac{V}{A} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha)}{\Gamma\left(\frac{1}{2}+\alpha\right)}\frac{V}{\tan\beta} \tag{12}$$

In the limit of vanising deadrise ($\beta \to 0$), $\alpha \to 1/2$ and then $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$ leading to the Wagner result. For limit of vertical deadrise ($\beta \to \pi/2$), $\alpha \to 1$ and then $\Gamma(1) = 1$ and $\Gamma(3/2) = \sqrt{\pi}/2$ leading to the von Karman solution (no run-up).

Results

Here the results are shown for two-dimesional wedge impact and initial results for three-dimensional impact.

2D Wedge impact The two-dimensional version of the high-order boundary element method is run for a constant deadrise wedge with several deadrise angles. In Figure 1, the vertical force is compared with the force reported in [4] as well as finite-volume CFD runs, and has similar results, but a straighter trendline. The trend on the force from the current method as the deadrise approaches zero is to the results of Wagner, as given in [2]. If the wetted surface expansion rate calculated in Eqn 12 is not used, labeled Generalized von Karman (GVK), the method approaches that of von Karman for diminishing deadrise. These trends are due to the fact that the wetted area expansion rate is specified in the present method, and are different from the other methods because those expansion rates are numerically calculated. The benefit of present method is that relatively large time-steps while retaining reasonable accuracy for the force. For example, the presented results were generated using only four time-steps.

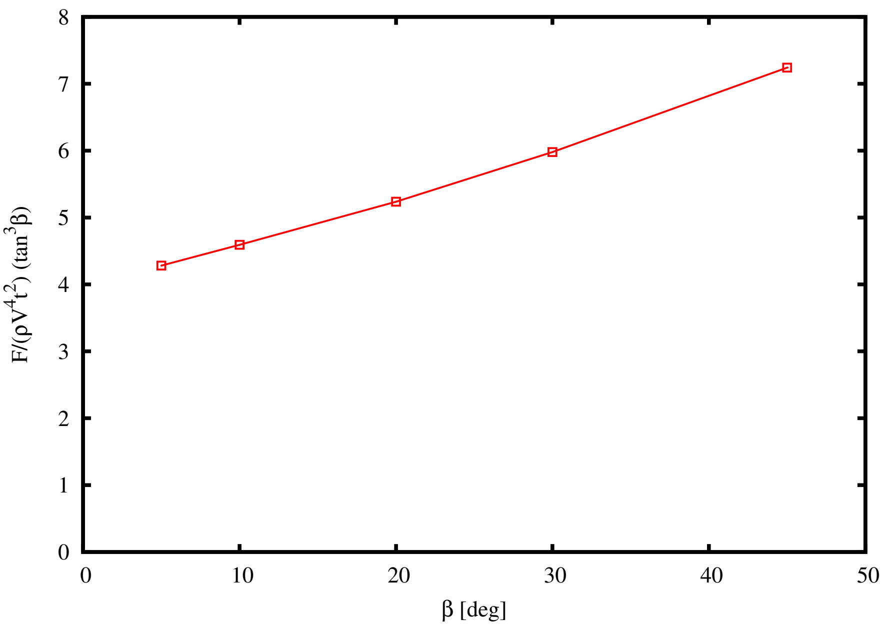

Figure 2: Non-dimensional force for constant velocity cone impact using a GVK method.

3D Impact Initial results for the constant velocity impact of a cone are shown in Figure 2, with only the generalized von Karman force computed currently with linear pressure. These results can be compared to analytical results described in [3] as the deadrise angle approaches zero. The heave added mass for a disc of radius a is $A_{33} = 4/3\rho a^3$, and the impact force is given by $F_3 = \partial/\partial t[A_{33}V]$. Shown in Figure 2 is the nondimensional force for several deadrise angles using a generalized von-Karman method. The trend is towards a non-dimensional force of 4 at zero deadrise, as is expected from the equation given above. CFD simulations will be performed for comparison as the three-dimensional generalized Wagner method is implemented, and comparisons will also be made to the numerical results of [3].

Acknowledgements

The authors would like to gratefully acknowledge the support of a grant from the US Office of Naval Research, Award # N00014-15-C-0132, under the technical direction of Ms. Kelly Cooper.

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