Symmetry in Multiple Body Calculations

M. H. Meylan[1] H. Wolgamot[2]

[1]School of Mathematical and Physical Sciences, University of Newcastle, Australia
[2]Faculty of Engineering, Computing and Mathematics, University of Western Australia, Australia
e-mail address: mike.meylan@newcastle.edu.au

Introduction

Symmetry plays a key role in mathematics and in the water wave context it can be exploited to simplify calculations. For our case of floating arrays the symmetry allows us to transform the problem to block diagonal form, and in some cases to diagonal form. As well as giving insights into the problem, this allows us to simplify our computations. The key results come from group theory and they say that if our matrices for added mass, damping etc. commute with the symmetries of the group then the solution must be decomposable into symmetries. This kind of decomposition is commonly applied to compact operators, for example to the eigenfunctions of a compact operator on a symmetric domain. We show that the same decomposition applies here and show how to derive the appropriate change of basis matrices. In particular we show how symmetry can be used to understand the motion of multiple heaving bodies.

Written in terms of matrices the solution in the time domain for multiple bodies is given by

$$\boldsymbol{\xi}(t) = \frac{2}{\pi}\int_0^\infty \mathrm{Re}\left\{\left(-s^2\mathbf{M}+\mathbf{C}-s^2\mathbf{A}(s)-\mathrm{i}s\mathbf{B}(s)\right)^{-1}\left(-\mathbf{M}-\mathbf{A}(s)-\frac{\mathrm{i}}{s}\mathbf{B}(s)\right)\mathrm{i}s\boldsymbol{\xi}(0)\right\}\cos(st)\,\mathrm{d}s, \quad (1)$$

(Meylan, 2014)

Complex Resonances

We define

$$\mathbf{D}(s) = -s^2\mathbf{A}(s) - \mathrm{i}s\mathbf{B}(s). \quad (2)$$

where $\mathbf{A}$ and $\mathbf{B}$ are the added mass and damping coefficients respectively. The equation for a complex resonance at ω_m is

$$\left(-\omega_m^2\mathbf{M} + \mathbf{C} + \mathbf{D}(\omega_m)\right)\mathbf{u}_m = 0. \quad (3)$$

and $\mathbf{u}_m$ is the resonance vector. Near the complex resonance at position ω_m with associated resonance vector $\mathbf{u}_m$ the solution can be approximated by

$$\hat{\boldsymbol{\xi}}(s) = \mathbf{u}_m \frac{\bar{\mathbf{u}}_m\left(-\mathbf{M}-\mathbf{A}(\omega_m)-\frac{\mathrm{i}}{s}\mathbf{B}(\omega_m)\right)\mathrm{i}\omega_m\boldsymbol{\xi}(0)}{(s-\omega_m)\bar{\mathbf{u}}_m\left(-2\omega_m\mathbf{M}+\mathbf{D}'(\omega_m)\right)\mathbf{u}_m}, \quad (4)$$

where $\mathbf{D}'$ is the derivative of $\mathbf{D}$ and $\bar{\mathbf{u}}_m$ is the row eigenvector.

Symmetry

Complex resonances are strongest when there is symmetry in the positions of the multiple bodies. We can exploit this symmetry if we introduce a transformation which is independent of frequency which reduces the added mass and damping matrices to block diagonal form. Group theory furnishes

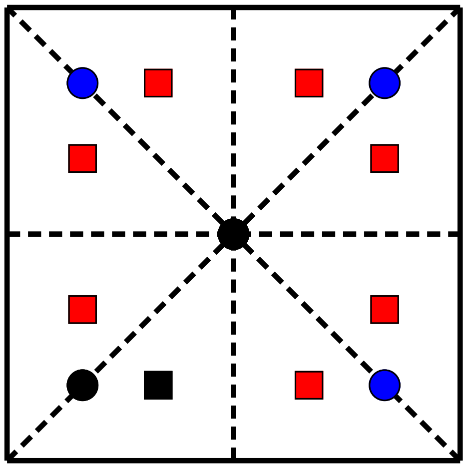

Figure 1: Diagram of the lines of symmetry of a square (shown as dashed lines). For a given point (black) we have shown the orbits (red/blue). We have shown the points with 1, 4 and 8 elements in the orbit.

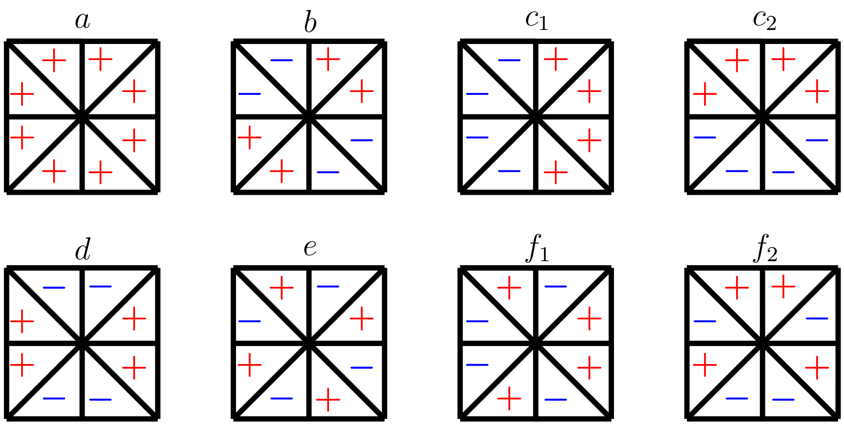

Figure 2: The eight symmetries for square arrays

us with a set of tools to accomplish this. A group is a set of symmetries or transformations (Dummit & Foote (2004)); here the elements of the group are rotations and reflections. The orbit of a group is the set of points that any point is mapped to under the action of the group. For a general point in a square there are 8 points in the orbit but for points which lie on lines of maximum symmetry there are only four points in the orbit, with the exception of the middle (invariant) point for which there is only one point in the orbit. This is shown schematically in figure 1. For our array of bodies to be symmetric we require that there is a cylinder at every point in the orbit of each cylinder.

It is straightforward to represent the elements of the group as matrices. For any of the arrangements above the matrices of added mass and damping will commute with these matrices. Figure 2 shows the irreducible subspaces of the symmetry group of a square. Each of these is mapped to itself (possibly with an overall change of sign) by all eight symmetries of a square. The elements c_1 and c_2 are a basis for an irreducible subspace of dimension two. In our decomposition this means that the block associated with each of these will be identical, so that a mode of oscillation in the form of c_1 will have the same frequency (and shape, rotated $\pi/2$) as that associated with mode c_2. The same is true for modes f_1 and f_2. The vector space formed by the invariants is a basis for the modes of oscillation of the square arrays, and will be used to form a transformation matrix, T.

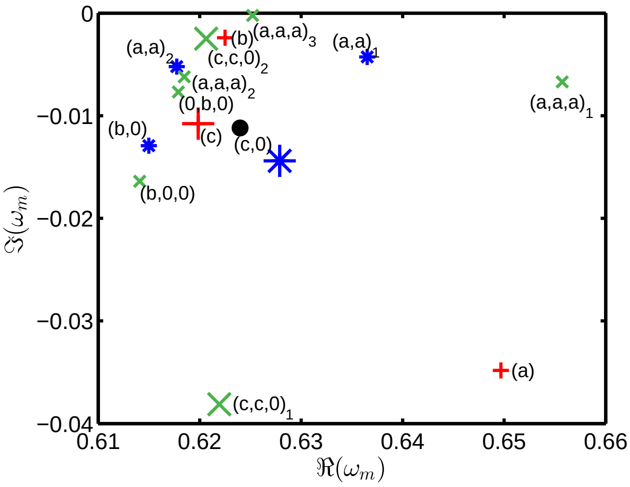

Figure 3: Complex resonance frequencies for 4, 5 and 9 cylinders for draft $d = 2$ for water of depth $h = 4$. Solid 1, crosses 4, star 5 and x 9 cylinders. The larger markers are doubles.

For the case of four cylinders in a square the T matrix is

$$\mathbf{T} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \tag{5}$$

where the 1/2 is to make the rows and column orthonomal. Note that the first row corresponds to invariant space a, the second to invariant space b, the third to invariant space c_1 and the fourth to invariant space c_2. We define a new basis $\hat{\boldsymbol{\zeta}}$

$$\hat{\boldsymbol{\xi}} = \mathbf{T}\hat{\boldsymbol{\zeta}} \tag{6}$$

which essentially means we are decomposing the motion into modes, with cylinders moving together or in pairs in opposite phase.

We present results for the motion for truncated cylinders using an eigenfunction matching method (Siddorn & Eatock Taylor, 2008). with non-dimensional gravity and the fluid density unity and fluid depth $h = 4$. The radius of the cylinders is unity. Figure 3 shows the complex resonances for the cases of 4, 5, and 9 cylinders. Figure 4 shows the real part of the complex resonance vectors for nine cylinders. We define a *Q factor* given by $Q_n = -|\omega_n|/2\mathrm{Im}\,\omega_n$

We can calculate the resonances for a perturbation of the array, breaking symmetry. We consider the nine cylinder case where we change the position of the cylinders by a small amount - a normally distributed random perturbation in both x and y (Figure 5). For the resonance closest to the real axis, every perturbation moves the resonance further away from the real axis.

References

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Siddorn, P & Eatock Taylor, R 2008 Diffraction and independent radiation by an array of floating cylinders. *Ocean Engineering* **35** (13), 1289–1303.

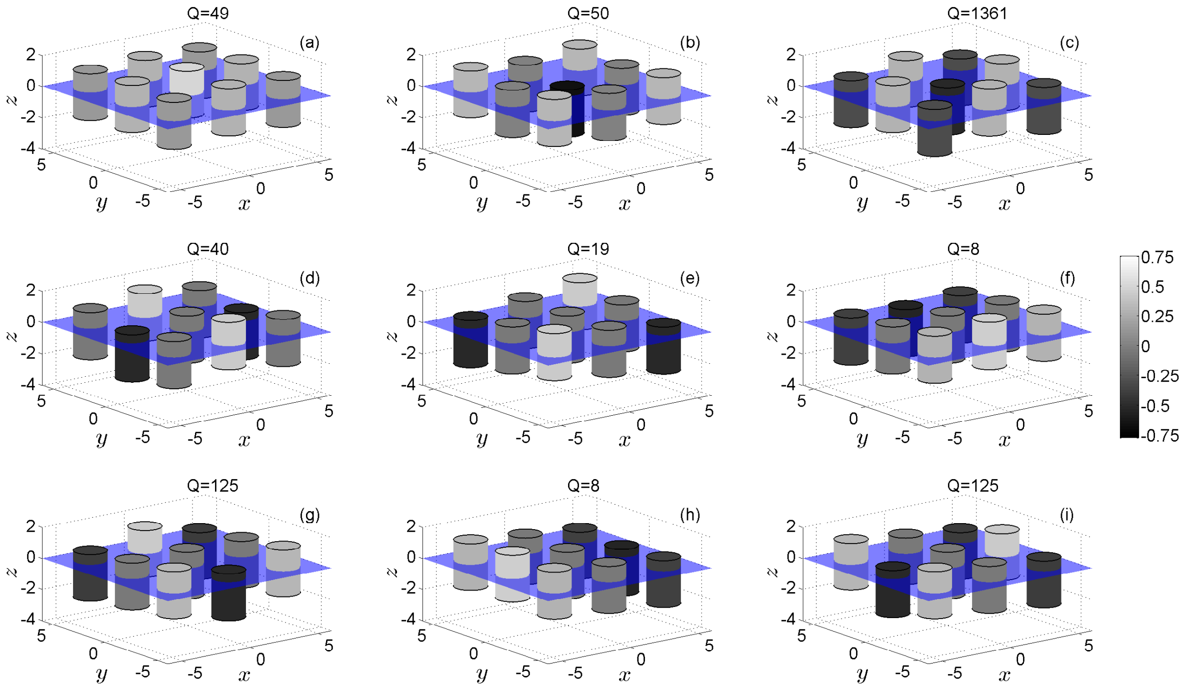

Figure 4: The real part of the nine complex resonance vectors for the nine cylinder array.

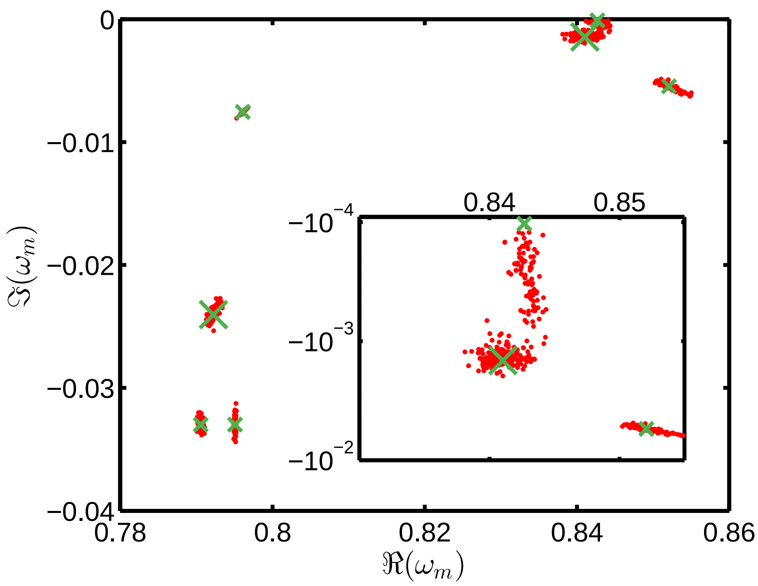

Figure 5: The case of nine cylinders with water depth $h = 4$ and draft 1. The crosses show the positions of the resonances as in figure 3 while the dots show configurations in which the cylinders are slightly perturbed from this mean position