Dispersion relations of waves generated by a traveling oscillating disturbance on a shear current

Yan Li* and Simen Å. Ellingsen
Department of Energy and Process Engineering,
Norwegian University of Science and Technology,
*Kolbjorn Hejes vei 2, N-7491 Trondheim, Norway**

Highlights

- The classical problem of a surface gravity waves generated by an oscillating and traveling surface disturbance are analysed in presence of shear current of uniform vorticity.
- The appropriate dispersion relations is derived, and graphical and explicit solutions are presented in deep waters. An expression for the Doppler resonance frequency is derived, showing strong asymmetry effects due to the shear. In particular, when the disturbance travels against the shear and $VS/g \geq 1$, the smallest resonance frequency can tend to zero [V: forward speed, S: vorticity, g: gravitational acceleration].
- A full account is given of the different wave solutions existing in different wave vector sectors of the far-field, as well as which waves may be found in front of the disturbance. Above a resonant frequency no waves exist within a sector of forward directions, as known also without shear current. Another sector also exists when $S > \omega_0$ (ω_0: oscillation frequency) within which certain far-field waves are "cut off" due to the presence of the shear current.

I. INTRODUCTION

The problem of waves generated by a pressure disturbance of oscillating strength travelling at constant velocity along the free water surface is one of fundamental theoretical interest [1] as well as practical importance to seakeeping of ships and in the study of wave loads on offshore vehicles[2]. It has been extensively studied for irrotational flow within the framework of linear wave theory since the first few decades of last century. The velocity potential was found in Refs.[3] in infinite water depth and extended to finite water depth in Refs. [4, 5]. Due to the Doppler effect, there exists a "resonant" value of the non-dimensional frequency $\tau = V\omega_0/g$ (ω_0 is the oscillating frequency, V is the forward speed) which is well known to be $1/4$ in an unbounded, deep water domain (c.f. Refs. [1, 3, 6–8]). This is the frequency at which forward propagating waves have zero group velocity relative to the wave source, and is associated with infinite wave amplitude in linear theory [9].

In particular, when $\tau < 1/4$, one wave solution was found in front of the moving source and three behind it, while two wave solutions were found for $\tau > 1/4$, both of which behind the source.

We solve the problem of a travelling, oscillating disturbance on a free surface when a sub-surface shear current of uniform vorticity is present. The introduction of shear greatly increases the richness of the problem. It is also of practical significance since waves and shear flows co-exist in many marine environments[10]. The profound effects of linearly depth-varying current on gravity waves in three dimensions have been demonstrated in the recent literature [11–13].

We begin by deriving the general solutions to the linearised problem, whence the dispersion relation is deduced and graphical solutions thereof are discussed. Subsequently, the dimensionless resonance frequency τ_{Res} is derived in deep waters, confirming the well-known value $1/4$ when shear is not present, while showing strong anisotropy with respect to direction of motion relative to the shear flow. The existence (or otherwise) of different far-field waves in different sectors of propagation directions depends on the subtle interplay of shear and oscillating frequency.

II. GENERAL SOLUTIONS

A wave-current system where waves are generated by a traveling oscillating disturbance and subsurface shear current linearly varies with water depth. The geometry of the problem is depicted in Fig. 1. As shown in Fig. 1, there is a free surface which, when undisturbed, is at $z = 0$, a shear current directed along the x direction given as $U(z) = Sz$ ($S \geqslant 0$ is the uniform vorticity), and the water depth is h. We assume incompressible flow and neglect effects of viscosity and surface tension. The resulting flow is solved to linear order in all perturbation quantities. The disturbance moves with constant speed $\mathbf{V}$ making an angle β with the x axis (hence with the current), while simultaneously oscillating harmonically with angular frequency ω_0. The velocity and pressure field we write

$$\mathbf{v} = (U(z) + \hat{u}, \hat{v}, \hat{w}); \quad P = \hat{p} - \rho g z, \tag{1}$$

in which all hatted quantities are small perturbations due to the waves; P and $\hat{p}$ are respectively the total and dynamic pressure; ρ is the fluid density and g is the acceleration of gravity.

The pressure disturbance depends on time through an overall factor $\exp(-\mathrm{i}\omega_0 t)$ (real part understood). We seek solutions which are purely oscillatory in time when seen from a coordinate system following the source. All quantities are thus presumed

*Electronic address: yan.li@ntnu.no

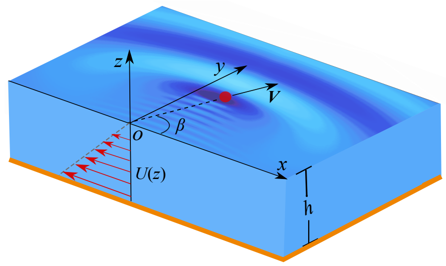


FIG. 1: The wave-current model

to depende only on position vector $\boldsymbol{\xi} = \mathbf{x} - \mathbf{V}t$ and on time through an overall factor $\exp(-\mathrm{i}\omega_0 t)$ only. The following form of solution is assumed,

$$[\hat{u}, \hat{v}, \hat{w}, \hat{p}](\boldsymbol{\xi}, z, t) = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\xi} - \omega_0 t)} [u, v, w, p](\mathbf{k}, z) \tag{2}$$

where $\mathbf{k} = (k_x, k_y) = (k\cos\theta, k\sin\theta)$ is the wave vector, conjugate to $\boldsymbol{\xi}$.

The 3-component Euler equations now read

$$\mathrm{i}(k_x U - \mathbf{k}\cdot\mathbf{V} - \omega_0)[u, v, w] + S[w, 0, 0] = -[\mathrm{i}k_x p, \mathrm{i}k_y p, p']/\rho \tag{3}$$

in which the prime denotes the derivative with respect to z, and the continuity equation is $\mathrm{i}k_x u + \mathrm{i}k_y v + w' = 0$.

Following steps similar to Refs. [11, 12] and applying the impermeability boundary condition at seabed, we obtain the solution of Eq.(3)

$$w(\mathbf{k}, z) = kA(\mathbf{k}) \sinh k(z+h), \tag{4}$$

from which u, v, p readily follow. $A(\mathbf{k})$ is an unknown variable, independent of z and t.

Similarly, we assume similar Fourier forms for the surface elevation, $\zeta(\boldsymbol{\xi}, t)$, and the applied external pressure distribution,

$$[\zeta(\boldsymbol{\xi}, t), \hat{p}_{\mathrm{ext}}(\boldsymbol{\xi}, t)] = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} [B(\mathbf{k}), p_{\mathrm{ext}}(\mathbf{k})] \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\xi} - \omega_0 t)}, \tag{5}$$

The linearized kinematic and dynamic boundary conditions at the free surface are now

$$kA \sinh kh = -\mathrm{i}\sigma_0(\mathbf{k}); \quad \mathrm{i}A(\mathbf{k})[\sigma_0(\mathbf{k}) \cosh kh + \frac{k_x S \sinh kh}{k}] - gB = \frac{p_{\mathrm{ext}}}{\rho} \tag{6}$$

with $\sigma_0(\mathbf{k}) = \mathbf{k}\cdot\mathbf{V} + \omega_0$ the Doppler shifted frequency. Eliminating $A(\mathbf{k})$ yields

$$B(\mathbf{k}) = -\frac{1}{\rho} \frac{k p_{\mathrm{ext}} \tanh kh}{[gk - k_x S \sigma_0(\mathbf{k})/k] \tanh kh - \sigma_0^2(\mathbf{k})}, \tag{7}$$

This is the general solution of surface waves which can be extended to deep waters by approaching kh to ∞. Ship waves, analysed in Refs. [11, 12], are given as the special case $\omega_0 = 0$.

III. DISPERSION RELATION AND RESONANCE FREQUENCY

We obtain the dispersion relation which permits nonzero solutions of the linearised equations of motion,

$$\sigma_0^2(\mathbf{k}) - [gk - k_x S \sigma_0(\mathbf{k})/k] \tanh kh = 0. \tag{8}$$

Nondimensionalising Eq. (8) with respect to length b and time $\sqrt{b/g}$ yields

$$\Omega_0 + K\mathrm{Fr}\cos\gamma = \Sigma_\pm(\mathbf{K}), \tag{9}$$

where $\Omega_0 = \omega_0\sqrt{b/g}$, $K = kb$, and $\mathrm{Fr} = V/\sqrt{gb}$ is the Froude number. The angle between $\mathbf{K}$ and $\mathbf{V}$ is $\gamma = \theta - \beta$, and the nondimensional intrinsic frequencies are

$$\Sigma_\pm(\mathbf{K}) = \pm\sqrt{K \tanh KH + (\tfrac{1}{2}\mathrm{Fr}_{sb}\cos\theta \tanh KH)^2} - \tfrac{1}{2}\mathrm{Fr}_{sb}\cos\theta\tanh KH, \tag{10}$$

where $H = h/b$ and $\mathrm{Fr}_{sb} = S\sqrt{b/g}$ is the "intrinsic shear Froude number".

In deep water, $KH \to \infty$, the dispersion relation permits explicit solutions

$$K_{C,E} = \frac{1 - \mathrm{Fr_s}\cos\theta\cos\gamma + 2\tau\cos\gamma + \sqrt{\Delta}}{2\mathrm{Fr}^2\cos^2\gamma}, \tag{11}$$

$$K_{B,D} = \frac{1 - \mathrm{Fr_s}\cos\theta\cos\gamma + 2\tau\cos\gamma - \sqrt{\Delta}}{2\mathrm{Fr}^2\cos^2\gamma}, \tag{12}$$

$$\Delta = (1 - \mathrm{Fr_s}\cos\gamma\cos\theta)^2 - 4\tau\cos\gamma, \tag{13}$$

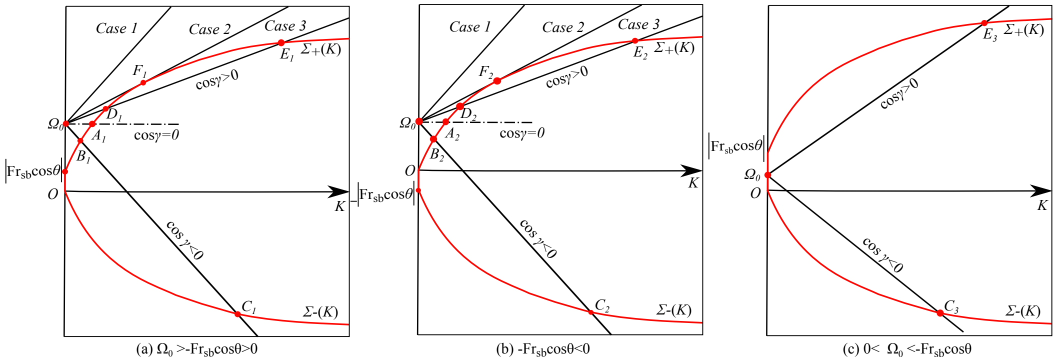


FIG. 2: Graphical solutions of dispersion relation in deep waters. In the figures, slope of the tangent of curve $\Sigma_\pm(K)$ denotes component of non-dimensional and intrinsic group velocity, $C_{gK}(K) = \partial\Sigma_\pm/\partial K$; Slope of the line connecting one point on curve $\Sigma_\pm(K)$ and origin denotes intrinsic phase velocity, $C(K) = \Sigma_\pm/K$; Slope of the straight line $\Omega_0 + K\mathrm{Fr}\cos\gamma$ is the projected component of non-dimensional moving speed of the source along direction of wave propagating - $\mathrm{Fr}\cos\gamma$.

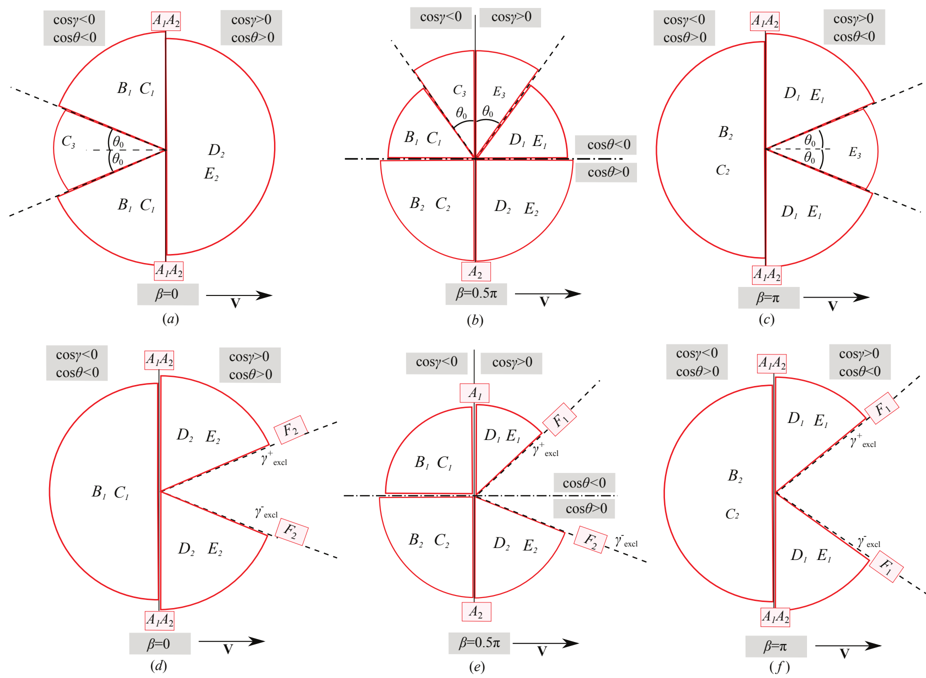


FIG. 3: Wave solutions varying with γ and θ based on dispersion relations depicted in Fig.(2) in deep waters. Subplots (a)-(c) show a sub-resonant, strong shear situation ($\tau < \tau_{\mathrm{Res,min}}, \omega < S$), subplots (d)-(f) show a super-resonant, weak-shear situation, $\tau > \tau_{\mathrm{Res,min}}, \omega > S$. γ_{excl} is the angle at which $\Delta(\gamma^\pm_{\mathrm{excl}}) = 0$.

where $\tau = \Omega_0\mathrm{Fr} = \omega_0 V/g$ is a nondimensional frequency, $\mathrm{Fr_s} = VS/g$ is the Shear Froude number based on length of g/S^2. Subscripts B, C, D and E correspond to graphical solutions shown in Fig. (2). Eq. (11) agrees with Ref. [1] when $\mathrm{Fr_s} = 0$. Solutions of the dispersion relation Eq. (8) are discussed further in Sec. IV.

A necessary criterion for a Doppler resonance to occur is that roots D and E of the dispersion relation (see Fig. 2) flow together in a single point F. In deep water this is clearly the case if the descriminant Δ is zero, which we can rearrange into criterion

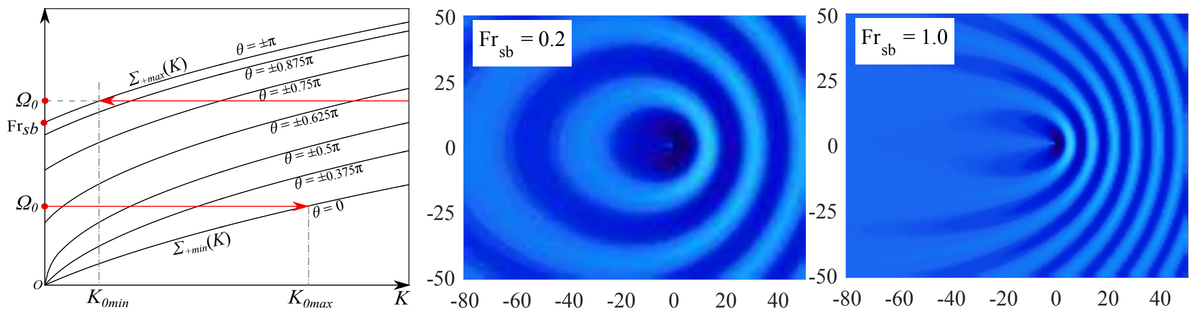
$$\tau = (1 - \mathrm{Fr_s}\cos(\gamma+\beta)\cos\gamma)^2/(4\cos\gamma) \tag{14}$$

while $\cos\gamma > 0$ should be understood to hold. The smallest resonance frequency $\tau_{\mathrm{Res,min}}$ (more than one resonance frequency could be found due to presence of the shear current) is the smallest value of τ for which this can occur, i.e.,

$$\tau_{\mathrm{Res,min}} = \min_{\gamma}\left\{(1 - \mathrm{Fr_s}\cos(\gamma+\beta)\cos\gamma)^2/(4\cos\gamma)\right\} \tag{15}$$

where the notation denotes that the minimum value is found with respect to γ in the sector $(-\pi/2, \pi/2)$. When $\mathrm{Fr_s} = 0$, $\tau_{\mathrm{Res}} = 1/4$ is obtained in agreement with, e.g., Refs.[1, 6, 7]. We term the situation as *sub-resonant* when $\tau < \tau_{\mathrm{Res,min}}$. Further analysis of resonances are found in Ref. [14].

IV. BRIEF DISCUSSIONS



(a)Graphical solutions at Fr = 0.

(b)Wave patterns at Fr = 0 and $\Omega_0 = 0.5$. Axes in units of b.

FIG. 4: Special case of zero source velocity

Although the deep water limit simplifies the dispersion relation and permits analytical solutions, it also introduces subtleties which the finite water case does not contain, via the phenomenon of cut-off, discussed in Ref. [13] for the 2D case. When KH tends to ∞, the graph of $\Sigma_\pm(K,\gamma)$ as a function of K obtains a vertical section from the origin to the value $-\mathrm{Fr}_{sb}\cos\theta$. The situation for infinite water is shown in Fig. 2.

As can be seen in Fig. 2, different wave solutions of Eq. (8) are possible in various situations, marked by A - F in the figure. Of these, only waves marked D can occur in front of a moving source because its group velocity exceeds the projection of V onto direction $\mathbf{k}$, while waves of type A are for a stationary source or wave components directed normal to $\mathbf{V}$.

The kind of waves available depends on two criteria; whether $\tau < \tau_{\mathrm{Res,min}}$ (sub-resonant) or not, and whether $\Omega_0 < \mathrm{Fr}_{sb}$ (or equivalently, $\omega_0 < S$) or not. We term the latter situation the "strong shear" case.

a. Weak shear. When $\Omega_0 > \mathrm{Fr}_{sb}$, only figures 2a and b are possible, since Ω_0 always exceeds $\mathrm{Fr}_{sb}|\cos\theta|$. When $\cos\gamma < 0$ (rearwards propagating waves), there are always two waves, of type B and C, and when $\mathrm{Fr}\cos\gamma = 0$, there is a single wave A. For forward propagating waves, $\cos\gamma > 0$, however, the number of waves depends on whether the situation is sub- or super-resonant. When $\tau < \tau_{\mathrm{Res}}$, only waves D and E are possible. When $\tau > \tau_{\mathrm{Res}}$, however, a sector of γ-values exists including $\gamma = 0$ in which there are no wave solutions. In this case the resonance criterion Eq. (14) will be satisfied for two values of γ, for which D and E waves flow together in a double root F.

b. Strong shear. When $\Omega_0 < \mathrm{Fr}_{sb}$ (or equivalently, $\omega_0 < S$), a third situation arises in propagation directions θ where $\Omega_0 < \mathrm{Fr}_{sb}|\cos\theta|$, that is to say, when $|\theta| < \theta_0 = \arccos(S/\omega_0)$. The situation is shown in Fig. 2c, and implies that waves of type A, B and D all tend to $K = 0$. These "infinite length" waves carry no energy and may simply be neglected [15]. This is the phenomenon referred to by Tyvand & Lepperød as "cut-off", whereby only one wave, of type C or E, may propagate along directions within the downstream sector $|\theta| < \theta_0$.

By way of example, a summary of which waves exist in which γ sectors is shown schematically for two different situations in Fig. 3. In each sector of the pie charts, the possible wave solutions are listed with reference to Fig. 2. Fig. 3a-c shows a strong-shear, sub-resonant situation, while Fig. 3d-e shows a weak-shear, super-resonant situation.

The special case of zero source velocity is exemplified in Fig. 4-a and 4-b, showing, respectively, graphical solutions to the dispersion relation and relief plots of the surface elevation for two different shear strengths. Unlike the recent work of Ellingsen & Tyvand for a submerged, oscillating point source[16], there is no critical layer contribution for a surface wave source.

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