Shallow Water Cloaking with Anisotropic Fluid

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Abstract

The coordinate transformation method for cloaking a body in electromagnetic and acoustic waves is extended to shallow water waves by noting variable correspondence in the governing equations. As a result, it is shown analytically that the fluid density should be anisotropic to achieve the cloaking of a body in shallow water. Numerical computations are also made using a commercial software COMSOL Multiphysics and good agreement with the cloaking in 2D electromagnetic and acoustic waves is indicated.

1. Introduction

Cloaking refers to a phenomenon that makes an object transparent, as its name suggests. The cloaking has originally been studied in electromagnetic wave fields since the work by Pendry *et al.* (2006) based on the coordinate transformation method. In recent years, this method is actively applied to other wave fields, such as acoustic waves (Cummer *et al.*, 2007), seismic waves (Brule *et al.*, 2014) and so on. In water waves, Porter (2011) showed that the coordinate transformation method could not be extended to water waves and studied the cloaking in terms of no scattered-wave energy by changing the local water depth using a mild-slope approximation. Newman (2014) and Iida *et al.* (2015) studied the cloaking with the idea of scattered-wave cancellation by surrounding a body with an array of smaller cylinders. However, the scattered-wave cancellation method cannot cloak a body itself. On the other hand, the coordinate transformation method can achieve it by creating a space where no wave exists. This method is based on invariance of the governing equations under the coordinate transformation. Although most other waves are not invariant under such transformation, Cummer *et al.* (2007) demonstrated that the acoustic and electromagnetic waves have equivalence in the equations written in the polar coordinates at least for 2D problems. Realizing the cloaking by the coordinate transformation implies that the medium should be anisotropic, but it is believed not to exist in the nature. Shurig *et al.* (2006) demonstrated a possibility to control such media (the permeability and permittivity in electromagnetic waves) and realized the cloaking by the so-called meta-material; this work has been applied to acoustic waves (Zigoneanu *et al.*, 2014) and water waves (Farhat *et al.*, 2008).

In this paper, we study the coordinate transformation method for cloaking a body in shallow water. It is shown that the shallow-water cloaking can be achieved by anisotropic fluid density. Furthermore, a transformation order (Zhang *et al.*, 2008) is introduced so as to reduce the degree of divergence. Then it is confirmed to be able to create a quiescent space around a body where waves cannot propagate. A linear potential flow is assumed with long-wave approximation and the software COMSOL Multiphysics is employed in numerical computations.

2. Theory

2.1 Coordinate transformation of Maxwell's equations

The cloaking was originally demonstrated for electromagnetic waves which are governed by a set of partial differential equations; these are called Maxwell's equations and described as

$$\left.\begin{aligned} \nabla\times\boldsymbol{E} &= -\mu_0\mu\,\frac{\partial\boldsymbol{H}}{\partial t} \\ \nabla\times\boldsymbol{H} &= +\varepsilon_0\varepsilon\,\frac{\partial\boldsymbol{E}}{\partial t} \end{aligned}\right\} \tag{1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ denote the electric and magnetic fields, respectively; μ and ε are the permeability and permittivity, respectively; and subscript 0 means quantities in vacuum. The coordinate transformation is based on invariance of Maxwell's equations. Let us consider the coordinate transformation between the original Cartesian mesh $\boldsymbol{x}$ and a distorted mesh $\boldsymbol{q}$. Then (1) can be transformed as

$$\left.\begin{aligned} \nabla_{\boldsymbol{q}}\times\widehat{\boldsymbol{E}} &= -\mu_0[\,\hat{\mu}\,]\,\frac{\partial\widehat{\boldsymbol{H}}}{\partial t} \\ \nabla_{\boldsymbol{q}}\times\widehat{\boldsymbol{H}} &= +\varepsilon_0[\,\hat{\varepsilon}\,]\,\frac{\partial\widehat{\boldsymbol{H}}}{\partial t} \end{aligned}\right\} \tag{2}$$

In order to keep the same form as (1) under the coordinate transformation, variables should take the following forms

$$\widehat{\boldsymbol{E}}(\boldsymbol{q}) = \boldsymbol{A}^T\boldsymbol{E}(\boldsymbol{x}), \quad \widehat{\boldsymbol{H}}(\boldsymbol{q}) = \boldsymbol{A}^T\boldsymbol{H}(\boldsymbol{x}) \tag{3}$$

$$[\,\hat{\mu}\,] = \frac{\boldsymbol{A}\,\mu\boldsymbol{A}^T}{|\boldsymbol{A}|}, \quad [\,\hat{\varepsilon}\,] = \frac{\boldsymbol{A}\,\varepsilon\boldsymbol{A}^T}{|\boldsymbol{A}|} \tag{4}$$

where $\boldsymbol{A}$ is the Jacobian transformation matrix between the original and transformed coordinate systems, and its coefficient can be written as

$$A_{ij} \equiv \frac{\partial q_i}{\partial x_j} \tag{5}$$

It should be noted that the permeability and permittivity become tensors as shown in (4). Therefore these parameters should possess anisotropic properties.

We consider the coordinate transformation to compress a space from a cylindrical region to a annular region so as to cloak an object in the annulus

$$r' = \frac{b-a}{b^n}r^n + a, \quad \theta' = \theta, \quad z' = z \tag{6}$$

where (r,θ,z) denotes the original cylindrical coordinate system; (r',θ',z') a new cylindrical coordinate system for cloaking; a and b are inner and outer radii of the cloaking region respectively, and n is the transformation order. Results of these transformations are shown in Figs. 1 and 2. Fig. 1 is the case of $n=1$ which Pendry *et al.* (2006) had demonstrated and Fig. 2 is the case of $n=1/3$ which Zhang *et al.* (2008) had proposed for an efficient transformation order. According to (4), tensor components in the cylindrical coordinate system can be obtained as

$$\left.\begin{aligned}\frac{\hat{\mu}_r}{\mu}&=\frac{\hat{\varepsilon}_r}{\varepsilon}=n\frac{r-a}{r}\\ \frac{\hat{\mu}_\theta}{\mu}&=\frac{\hat{\varepsilon}_\theta}{\varepsilon}=\frac{1}{n}\frac{r}{r-a}\\ \frac{\hat{\mu}_z}{\mu}&=\frac{\hat{\varepsilon}_z}{\varepsilon}=\frac{1}{n}\frac{b^2}{(b-a)^{2/n}}\frac{(r-a)^{2/n-1}}{r}\end{aligned}\right\}\qquad(7)$$

Here we have rewritten r' as r. Eq. (7) indicates that when $r=a$, the θ-component of permeability and permittivity diverges. A smaller value of transformation order n (such as $n=1/3$) can practically reduce this divergence feature as shown in Fig. 2. The impedance at $r=b$ is equal to 1.0, which matches with the outer region.

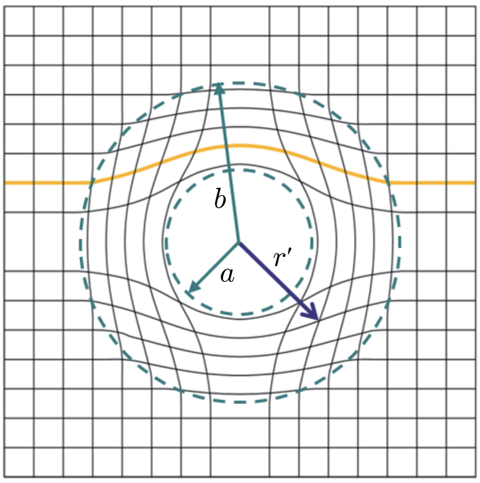

Fig. 1 Cloaking coordinate system with $n=1$.

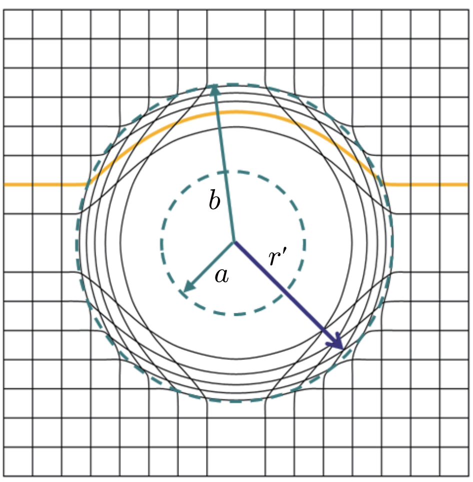

Fig. 2 Cloaking coordinate system with $n=1/3$.

2.2 Equivalence to shallow-water equations

If the coordinate transformation method can be applied to other waves, the cloaking can also be achieved in those waves. However, this is normally not the case in most real waves. In linear water waves, Porter (2011) confirmed that the governing equations are scalar and hence not invariant under the coordinate transformation. On the other hand, Cummer *et al.* (2007) demonstrated the cloaking in 2D acoustic waves by noting the equivalence between acoustic and electromagnetic waves; that is, they showed variable equivalence in the equations described in the polar coordinate system. We can show that the same idea can be extended to 2D water waves especially under the long-wave approximation known as shallow water.

First, let us consider expressions of 2D Maxwell's equations given by (1) in the polar coordinate system. By assuming z-invariant transverse electric waves and time harmonic variation (with circular frequency ω), Maxwell's equations are written as

$$\left.\begin{aligned}i\omega\,\mu_r(-H_r)&=-\frac{1}{r}\frac{\partial(-E_z)}{\partial\theta}\\ i\omega\,\mu_\theta\,H_\theta&=-\frac{\partial(-E_z)}{\partial r}\\ i\omega\varepsilon_z(-E_z)&=-\frac{1}{r}\frac{\partial(rH_\theta)}{\partial r}-\frac{1}{r}\frac{\partial(-H_r)}{\partial\theta}\end{aligned}\right\}\qquad(8)$$

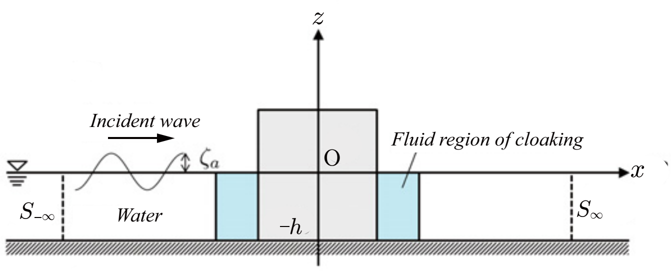

Fig. 3 x-z plane of shallow water.

In the same way, water-wave equations will be considered with cylindrical coordinate system. Assuming inviscid fluid with irrotational motion and linear long-wave approximation (the water depth h is sufficiently smaller than the wave length λ), we can write the Bernoulli's pressure equation and Euler's equation in the form

$$\left.\begin{aligned}\frac{\partial P}{\partial t}&=-\rho c^2\,\nabla\cdot\boldsymbol{u}\\ \frac{\partial\boldsymbol{u}}{\partial t}&=-\frac{1}{\rho}\nabla P\end{aligned}\right\}\qquad(9)$$

where P is the pressure; ρ the fluid density; c the phase velocity equal to $\sqrt{gh}$; and $\boldsymbol{u}$ the velocity vector. They can be expressed in the cylindrical polar coordinate system as

$$\left.\begin{aligned}i\omega\,\rho_\theta\,u_\theta&=-\frac{1}{r}\frac{\partial P}{\partial\theta}\\ i\omega\,\rho_r\,u_r&=-\frac{\partial P}{\partial r}\\ i\omega\frac{1}{\rho_z}P&=-\frac{1}{r}\frac{\partial(ru_r)}{\partial r}-\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\end{aligned}\right\}\qquad(10)$$

where

$$\rho_z\equiv\rho c^2\qquad(11)$$

Comparison of (10) with (8) can find the following correspondence:

$$\begin{aligned}&\big[-E_z, H_\theta, -H_r, \mu_\theta, \mu_r, \varepsilon_z\big]\\&\longleftrightarrow \big[P, u_r, u_\theta, \rho_r, \rho_\theta, 1/\rho_z\big]\end{aligned} \tag{12}$$

Therefore if we assume an anisotropic fluid density as in (7), the tensor components of fluid density may be written as

$$\left.\begin{aligned}\rho_r &= \rho_0\,\frac{1}{n}\frac{r}{r-a}\\ \rho_\theta &= \rho_0\, n\frac{r-a}{r}\\ \rho_z &= \rho_0 c_0^2\, n\frac{(b-a)^{2/n}}{b^2}\frac{r}{(r-a)^{2/n-1}}\end{aligned}\right\} \tag{13}$$

Here ρ_0 is the fluid density outside of the cloaking region.

Shallow water is governed by the 2D Helmholtz equation which satisfies the continuity equation, the Euler's equation, and the free surface and bottom boundary conditions. It is written as

$$\nabla\cdot\Big(-\rho_0^{-1}\nabla\phi\Big)-\frac{\omega^2}{\rho_0 c_0^2}\,\phi=0 \qquad \text{on the general fluid region} \tag{14}$$

Eq. (14) must be modified on the cloaking region in terms of the density tensor as follows:

$$\nabla\cdot\Big(-[\rho^{-1}]\,\nabla\phi\Big)-\frac{\omega^2}{\rho_z}\,\phi=0 \qquad \text{on the cloaking fluid region} \tag{15}$$

Note that (15) is an expression in the Cartesian coordinate system even though (13) is written in the cylindrical coordinate system. In practical computations, it is necessary to rewrite (15) with the coordinate transformation matrix.

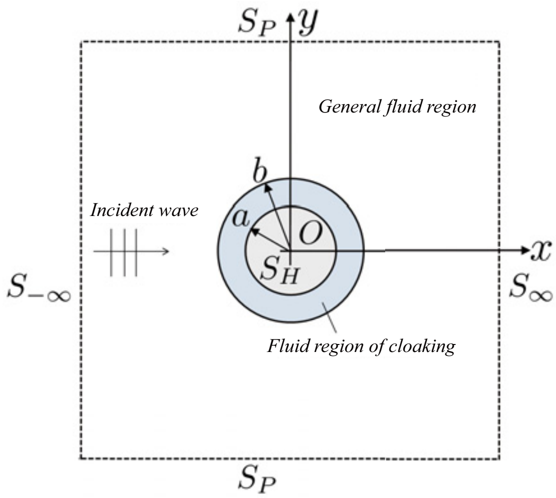

Fig. 4 x-y plane of shallow water.

3. Results and Discussions

In order to solve (14) and (15) simultaneously for a case shown in Fig. 4, a commercial software, COMSOL Multiphysics, was employed. COMSOL Multiphysics is a general-purpose software platform, based on the finite element method, for modelling and simulating physics-based problems and hence capable of treating anisotropic property. Since an anisotropic fluid density should be considered, this software is suitable to simulate the cloaking phenomenon in the present study. In numerical computations, we fix radii of inner and outer cloaking regions such that $a = 1.0$ and $b = 2.0$, wave length $\lambda = 1.5$, and water depth $h = 0.01$. (All quantities are normalized with fluid density of general region ρ_0, incident wave amplitude ζ_a, and radius of inner cloaking region a.) A cylinder with radius a is bottom-mounted as seen in Fig. 3.

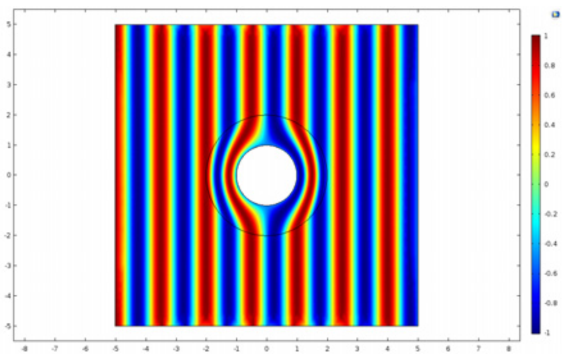

Fig. 5 Spatial wave pattern generated by transformation order $n = 1$, cloaking a bottom-mounted cylinder.

Figure 5 is a result with transformation order $n = 1$. Obviously the incident wave is not scattered by the cylinder and transmits downstream with same form as that of initial incident wave, unlike the situation of a cylinder in the fluid with uniform density. This result is very similar to the results of electromagnetic cloaking shown by Pendry *et al.* (2006) and of acoustic cloaking by Cummer *et al.* (2007). Therefore we may say that the coordinate transformation method can be extended successfully to shallow water.

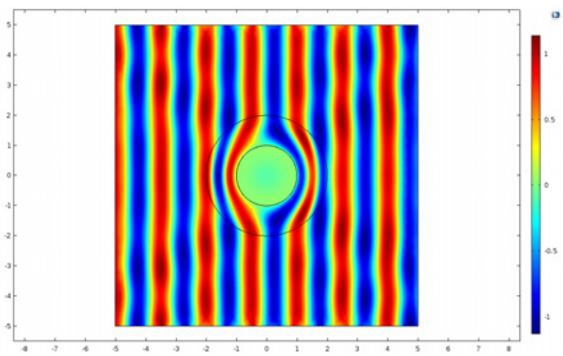

Fig. 6 Spatial wave pattern in the shallow water by transformation order $n = 1$ without the cylinder.

Figure 6 is also a result with transformation order $n = 1$, but the bottom-mounted cylinder is taken out from the inner region. Instead, the inner region is filled with constant density ρ_0. Mathematically the coordinate transformation method should work even for a case without the body and thus no scattered waves may be expected. However, Fig. 6 shows obviously wave reflection. This is because ρ_r and ρ_z diverge when $r = a$. Therefore the numerical accuracy around $r = a$ is not sufficient and consequently the entire system of the problem cannot be solved precisely.

In order to avoid this inconvenience, another choice of transformation order is helpful. As shown in Fig. 2, the case of $n = 1/3$ can shift the region around $r = a$ to outward of the cloaking region and hence the diverging feature can be reduced. Computed results are shown in Fig. 7, from which we can see few waves propagating into the inside domain.

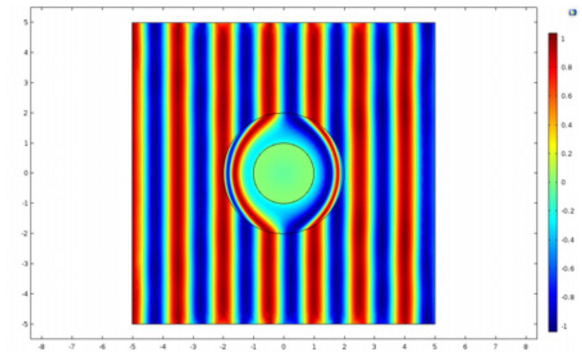

Fig. 7 Spatial wave pattern generated by transformation order $n = 1/3$ without the cylinder.

In this case, structures of any shape can be put in the interior region, such as a T-shape object, because the object is not affected by waves, as shown in Fig. 8.

The cloaking is a phenomenon to make an object transparent. This description includes two properties. First is to make incident waves propagate without being scattered by an object. Therefore we can observe waves (light, acoustic or water) behind a cloaking device. Second is to avoid wave invasion into the inside cloaking region. Therefore a space around the body cloaked by the device must be quiescent. First property is satisfied by not only the coordinate transformation method but also the scattered-wave cancellation method (such as Newman, 2014). If we focus ourselves on the wave itself, the first property is enough. However, most engineering interests are not waves but floating bodies on the free surface. The scattered-wave cancellation method can reduce the wave drift force acting on floating bodies, but other hydrodynamic forces on bodies are not zero. This is because the scattered-wave cancellation method does not satisfy the second property. In contrast, the coordinate transformation method can satisfy both properties. Thus, once it is put in the cloaking region as in Fig. 8, no forces act on that body. Because of this, the present study holds a practical application to operate offshore structures safely regardless of sea conditions.

From the results above, we can see that the cloaking can be realized with an anisotropic fluid. Although the present study is just a numerical study, we note that experimental possibility is already demonstrated by Farhat *et al.* (2008).

4. Conclusions

It has been demonstrated that the idea of cloaking in electromagnetic waves using the coordinate transformation method can be extended to the case of shallow-water waves, by noting the correspondence of variables in the governing equations between water waves and electromagnetic waves. As a consequence, the fluid density must be anisotropic to achieve the cloaking of a body in shallow water.

Numerical computations have also been made using a commercial software COMSOL Multiphysics, and good agreement with the cloaking in 2D electromagnetic and acoustic waves has been indicated. Furthermore, it has been

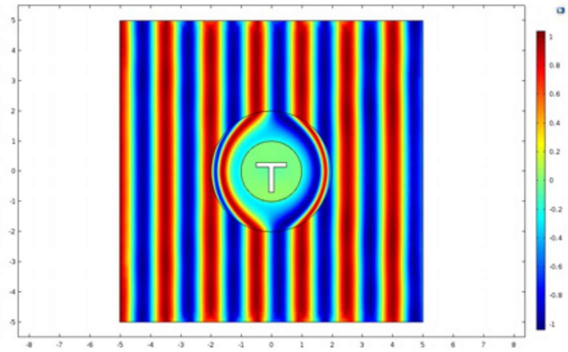

Fig. 8 Spatial wave pattern in cloaking by transformation order $n = 1/3$ with a T-shape object put inside interior quiescent region.

suggested that the wave reflection and invasion observed due to diverging feature of anisotropic density can be mitigated by selecting a smaller value for the transformation order, in particular for the case of $n = 1/3$ no wave scattering from the inside region and creation of a quiescent interior region are confirmed. Once this situation could be realized, any structures can be put in a cloaking region without any influence from the waves.

References

Brule, S., Javelaud, E. H., Enoch, S., & Guenneau, S. (2014). "Experiments on seismic metamaterials: Molding surface waves", *Physical review letters*, 112(13), 133901.

Cummer, S. A., & Schurig, D. (2007). "One path to acoustic cloaking", *New Journal of Physics*, 9(3), 45.

Farhat, M., Enoch, S., Guenneau, S., & Movchan, A. B. (2008). "Broadband cylindrical acoustic cloak for linear surface waves in a fluid", *Physical review letters*, 101(13), 134501.

Iida, T., Kashiwagi, M., & Miki, M. (2015) "Wave Pattern in Cloaking Phenomenon around a Body Surrounded by Multiple Vertical Circular Cylinders", *to appear International Journal of Offshore and Polar Engineering.*

Newman, J. N. (2014). "Cloaking a circular cylinder in water waves", *European Journal of Mechanics-B/Fluids*, 47, 145-150.

Pendry, J. B., Schurig, D., & Smith, D. R. (2006). "Controlling electromagnetic fields", *Science*, 312(5781), 1780-1782.

Porter, R. (2011). "Cloaking of a cylinder in waves", *In 26th International Workshop on Water Waves and Floating Bodies*, Athens.

Schurig, D., Mock, J. J., Justice, B. J., Cummer, S. A., Pendry, J. B., Starr, A. F., & Smith, D. R. (2006). "Metamaterial electromagnetic cloak at microwave frequencies", *Science*, 314(5801), 977-980.

Zhang, L., Yan, M., & Qiu, M. (2008). "The effect of transformation order on the invisibility performance of a practical cylindrical cloak", *Journal of Optics A: Pure and Applied Optics*, 10(9), 95001.

Zigoneanu, L., Popa, B. I., & Cummer, S. A. (2014). "Three-dimensional broadband omnidirectional acoustic ground cloak", *Nature materials*, 13(4), 352-355.