Mathematical modelling of the WITT wave energy converter

S. Crowley & R. Porter
sarah.crowley@bris.ac.uk, richard.porter@bris.ac.uk
University of Bristol, University Walk, Bristol, BS8 1TW, UK.

1 Introduction

The WITT (see http://www.witt-energy.com/) is a mechanical device for converting motion into electrical energy. It is comprised of a heavy compound pendulum connected through a gearbox in such a way that its rotary motion about either of two perpendicular horizontal axes is transferred to a single unidirectional output from which the energy of motion can subsequently be harvested.

This paper outlines some of the work being carried out at Bristol University in developing a mathematical model which can be used to assess the feasibility of using a WITT housed within a sealed hull in the ocean to convert ocean wave energy into electrical energy. The idea of using a mechanical device with counterweights inside a sealed hull to absorb wave energy is not new, for example the SEAREV (see, Cordonnier et al. (2015)) and the Wello (see, http://www.wello.eu/en/) use a similar principle.

The analysis that follows is novel, but similarities can be seen with a previous study of a fully submerged, horizontal cylindrical wave energy converter (WEC) constrained to move in pitch and surge with an internal pendulum (assumed to operate in a similar manner to the WITT), previously studied in Crowley, Porter & Evans (2013).

2 Device description

This paper addresses the modelling of a specific embodiment of the WITT WEC (WWEC) in which a WITT device is placed within a semi-immersed sealed spherical hull of radius a, which is free to move in heave, Z, surge, X, and pitch, Θ but is restrained by a four-point catenary mooring system in which heavy chains connect the hull of the WEC to the sea floor. This mooring system has the obvious practical role of preventing the WWEC from drifting away from its installation site. It also supplies spring restoring forces to the device when it moves in response to waves.

Internal to the sphere is a compound pendulum of natural length l, rotationally symmetric about the vertical and forming an annular sector in cross-section. The pendulum is allowed to rotate in pitch about the central horizontal axis of the sphere, making an angle θ_p with respect to the vertical axis. Power is taken off via a linear damper which acts in proportion to the relative angular velocity of the pendulum with respect to the pitch rotation of the sphere. A point mass is placed on the lower vertical axis of the spherical shell, representing the combined effect of ballast, the WITT gearbox and power-take off (PTO) machinery such that the centre of gravity of the hull and the point mass lies some distance below the centre of the sphere. Resolving the vertical forces on the sphere and mooring lines determines the mass of ballast required for the device to be semi-submerged when in equilibrium.

3 Mathematical modelling

The mathematical model of the device described in Section 2 can be broken down into three components: (i) the hydrodynamic response of a sphere in waves; (ii) the mathematical model of the mooring system; and (iii) the dynamics of the coupled system.

A simple mathematical model of a four-point mooring system has been developed. Each of the mooring lines are modelled to represent a catenary chain through the placement of a single point mass an arbitrary distance along a light line connecting the spherical hull to the sea floor, see Fig. 1.

Under the small amplitude assumption, the dynamic mooring restoring forces, $\boldsymbol{X}_m$, in the three assumed directions of surge, heave and pitch are expressed in terms of a 3×3 symmetric matrix K of linear spring constants.

As in Crowley, Porter & Evans (2013), the equations of motion for the WWEC system are derived from the Euler-Lagrange equations. Generalised coordinates X, Z, Θ are used to represent the surge, heave and pitch displacements of the sphere and θ_p the angle of pitch of the pendulums with respect to that of the vertical, as shown in Fig. 1.

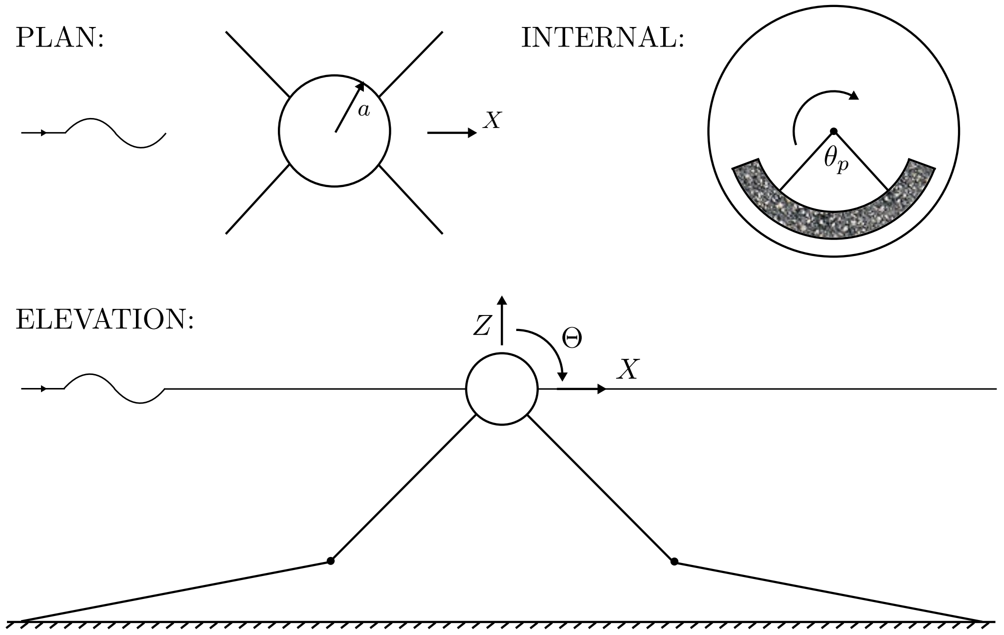

FIGURE 1: Plan, elevation and internal sketch of the system, showing directions of modes of motion.

Assuming small amplitude excursions, and considering time-harmonic motion of angular frequency ω such that

$$(\dot{X}, \dot{Z}, a\dot{\Theta}, l\dot{\theta}_p) = \mathrm{Re}\{(U_x, U_z, U_\Theta, u)\mathrm{e}^{-\mathrm{i}\omega t}\}, \quad (1)$$

we linearise the resulting equations. After some rearranging into a form indicative of Newton's Law, the equation of motion of the system is given in matrix/vector form by,

$$-\mathrm{i}\omega \mathsf{M}\boldsymbol{U} = \boldsymbol{X}_w - \frac{\mathrm{i}}{\omega}\left(\mathsf{C} + \mathsf{K}\right)\boldsymbol{U} + \boldsymbol{X}_e. \quad (2)$$

The complex velocity vector $\boldsymbol{U}$ is given by, $\boldsymbol{U} = (U_x, U_z, U_\Theta, v)^T$, where we have introduced a change of variables from u to v via

$$v = u - lU/a, \quad (3)$$

such that v represents the relative angular velocity of the pendulum and the sphere.

The vector $\boldsymbol{X}_w$ represents the total wave forces on the sphere, calculated following Hulme (1982) assuming water of infinite depth. The matrix M is symmetric and contains terms relating to the mass and rotational inertia of the sphere and internal pendulum. Restoring forces due to the buoyancy of the system and the mooring lines are given by C and K respectively. The elements of the matrices M, C and K may all determined by the physical properties of the system.

Finally, external forces imposed on the system by the PTO mechanism are given by,

$$\boldsymbol{X}_e = -\gamma \mathsf{G}\boldsymbol{U}, \quad (4)$$

where γ is the damping PTO parameter and, $\mathsf{G}_{ij} = \delta_{i4}\delta_{j4}$, for $i, j = 1, 2, 3, 4$, where δ_{ij} is the Kronecker delta function.

4 Power calculation

We calculate the total mean power (time averaged over a period) absorbed by the device. This is given by

$$W = \tfrac{1}{2}\mathrm{Re}\{\boldsymbol{X}_w^*\boldsymbol{U}\} = -\tfrac{1}{2}\mathrm{Re}\{\boldsymbol{X}_e^*\boldsymbol{U}\}, \quad (5)$$

where * denotes the complex conjugate transpose, and where (2) and the fact that the elements of M, K and C are all real has been used.

The single-frequency wave exciting forces on the sphere, $\boldsymbol{X}_w$, can be decomposed using linearity in the usual way into vectors containing forces due to the diffraction by a motionless sphere, $\boldsymbol{X}_s(\beta)$, and forces due to radiation,

$$\boldsymbol{X}_w = \boldsymbol{X}_s(\beta) + (\mathrm{i}\omega \mathsf{A} - \mathsf{B})\boldsymbol{U}, \quad (6)$$

where β is the angle of the incident wave and A and B contain the frequency dependent added mass and radiation damping matrices. Only $\mathsf{A}_{11}, \mathsf{A}_{22}$ and $\mathsf{B}_{11}, \mathsf{B}_{22}$ are non-zero, relating to the added mass and damping coefficients in surge and heave respectively. Similarly, $\boldsymbol{X}_s(\beta) = (X_s^S, X_s^H, 0, 0)$ as there are no wave exciting forces in pitch as the device shell is spherical.

Using equation (6) and (4), the equation of motion (2) can be rewritten as,

$$(\mathsf{Z} + \gamma \mathsf{G})\,\boldsymbol{U} = \boldsymbol{X}_s, \quad (7)$$

where,

$$\mathsf{Z} \equiv \mathsf{B} - \mathrm{i}\omega\left(\mathsf{A} + \mathsf{M} - \omega^{-2}(\mathsf{C} + \mathsf{K})\right). \quad (8)$$

Using equation (7) and (4), in (5) we write the power absorbed by the device as,

$$W = \tfrac{1}{2}\gamma \boldsymbol{X}_s^* \mathsf{E}^* \mathsf{G}\mathsf{E}\boldsymbol{X}_s, \quad (9)$$

where the components of the 4×4 matrix

$$E = (Z + \gamma G)^{-1}, \tag{10}$$

can be calculated explicitly.

We continue by assuming the elements of Z in (8) are assigned Z_{ij} and E in (10) are assigned E_{ij} for $i, j = 1, 2, 3, 4$.

With some persistence it can be shown that

$$W = \frac{1}{4}\frac{\left|X_s^S E_{41} + X_s^H E_{42}\right|^2}{|E_{44}|^2\left(|Y| + \mathrm{Re}\{Y\}\right)} \times \left(1 - \frac{(\gamma - |Y|)^2}{|\gamma + Y|^2}\right), \tag{11}$$

where we have set,

$$Y = Z_{44} + \frac{Z_{14}E_{14} + Z_{24}E_{24} + Z_{34}E_{34}}{E_{44}}, \tag{12}$$

such that,

$$\mathrm{Re}\{Y\} = \frac{B_{11}|E_{41}|^2 + B_{22}|E_{42}|^2}{|E_{44}|^2}. \tag{13}$$

At this point it is worth noting that the PTO parameter γ only appears where it is seen explicitly in the final bracket of equation (11).

5 Capture width, maximum and optimal power

The performance of a particular configuration will be considered in terms of the device capture width, l, which provides a measure of the mean power absorbed by the device, W, to the mean power per unit crest length of the waves incident upon the WEC, W_{inc}. In other words,

$$l(T, \beta) = \frac{W}{W_{inc}}, \tag{14}$$

defines the capture width for fixed PTO parameter γ, being the equivalent length of wave crest of incident wave power absorbed by the device.

For a spherical device moving in heave, surge and pitch it can be shown using (5) and (6) that the maximum power absorbed by the device will be given by,

$$W_{max} = \frac{\left|X_s^S\right|^2}{8B_{11}} + \frac{\left|X_s^H\right|^2}{8B_{22}}, \tag{15}$$

when $U_x = X_s^S/2B_{11}$ and $U_z = X_s^H/2B_{22}$ simultaneously.

Note again here that as the device is spherical, there are no wave exciting forces due to rotation. If we were instead dealing with a floating, vertical, cylindrical device there would also be a contribution to the maximum power from the pitch wave exciting forces.

It is well known that an axisymmetric, spherical device free to move in surge, heave and pitch has a theoretical maximum capture factor,

$$l_{max} = \frac{W_{max}}{W_{inc}} = \frac{3\Lambda}{2\pi}, \tag{16}$$

where Λ is the wavelength of the incident wave.

From equation (11) it can be seen that equation (14) provides an upper bound on the achievable device capture width, l_{opt}, when the PTO parameter is tuned such that $\gamma = |Y|$.

There is some strong evidence to suggest that the two conditions required for the optimum capture width to be equal to the theoretical maximum capture width are incompatible and cannot be satisfied.

If however, the spherical device were constrained to move in *either* surge *or* heave, whether in conjunction with pitch or not, then the prefactor in the expression for maximum capture width (16) would be reduced to 1 or 1/2 respectively. In such a case we could define an optimum capture width, achieved when $\gamma = |Y|$, equal to

$$l_{opt} = l_{max}\frac{2B_{ii}|E_{4i}|^2}{|E_{44}|^2\left(|Y| + \mathrm{Re}\{Y\}\right)}, \quad \text{for } i = 1, 2. \tag{17}$$

Furthermore, when $\mathrm{Im}\{Y\} = 0$ it is straight forward to show that $\mathrm{Re}\{Y\} = B_{ii}|E_{4i}/E_{44}|^2$ such that at this frequency the power *would* reach its maximum and $l = l_{opt} = l_{max}$.

6 Optimisation & Results

Ultimately we wish to predict and optimise the device performance over a wave energy spectrum for a given device test site. We have considered the Billia Croo EMEC test site, on the western edge of the Orkney mainland. For context, the EMEC site has an annual average wave power of 21kW/m and an average water of depth 50m, justifying our earlier deep water assumptions for the calculation of the hydrodynamic coefficients.

A scatter diagram of probabilities of expected sea states can be found in Neilsen (2010), allowing us to define a function $P(H_s, T_p)$ to be the joint probability of the occurrence of a pair of parameter values describing a particular sea state, where H_s is the significant wave height and T_p the peak wave period.

We employ the two parameter spectrum developed by Bretschneider (1959), and using the probability function $P(H_s, T_p)$ along with a function $G(\theta)$

to describe the angular spread of the energy density of the incident wave field, define a modified spectrum, $\tilde{S}(T,\theta)$.

The total mean power absorbed by a device of width $2a$ is then

$$\overline{W} = \rho g \int_{-\pi}^{\pi} \int_{0}^{\infty} c_g(T)\tilde{S}(T,\theta) l(T,\theta) T^{-2}\,\mathrm{d}T\,\mathrm{d}\theta, \tag{18}$$

where $l(T,\theta)$ is the capture width of the device given by (14), expressed here as a function of period, T, and the angle of incidence.

We can define a dimensionless *mean* capture factor,

$$\bar{l} = \frac{\overline{W}}{\overline{W}_{inc} 2a}, \tag{19}$$

which describes the mean proportion of incident wave power absorbed per unit width of the device. This now allows us to simply optimise the total mean capture width, taking into account the varying spectral wave energy density across wave period.

Alternative measures of relative device performance might consider the mean capture width per device submerged volume or surface area, which for a spherical device might be considered more appropriate.

With many free parameters in this problem, we employ a numerical optimiser from the NAG library (E04JYF) to determine the parameter values which maximise the mean capture factor, $\bar{l}$, over a given wave energy spectrum. In order to reduce the numerical effort required, a number of parameters are fixed: for example, the density of the pendulums are set to that of concrete and the shell is assumed to be of thickness 20mm and to be made of steel. Further numerical bounds are also imposed on some of the parameter values in order to ensure that the final optimised configuration is physically realisable.

In Fig. 2, the mean capture factor for a series of devices of increasing diameter is plotted. There appears to be an approximately linear increase in mean capture factor with device diameter, up to diameters of around 12m. This suggests there is little to gain in considering devices larger than approximately 12-15m in diameter.

In Fig. 3 the maximum, optimum and actual capture width ratio is plotted for a device with 15m diameter, a capture width ratio greater than unity is achieved across wave periods of approximately 7-12s. The four peaks in l_{opt} in Fig. 3 illustrate multiple device resonances. In a model sea state, representative of the EMEC site, this device is predicted to output roughly 240kW, with a mean capture factor of 0.77. The total device mass is just over 600 tonnes including a pendulum weighing close to 400 tonnes with the assumption that the mooring mass is 10% of the mass of the device, around 60 tonnes.

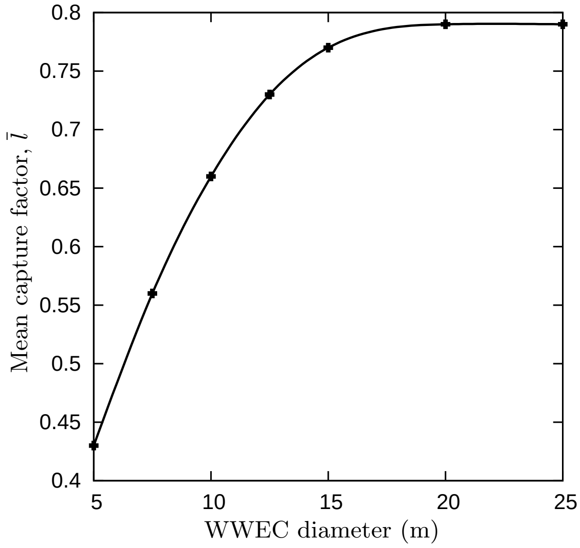

FIGURE 2: Mean capture factor against device size.

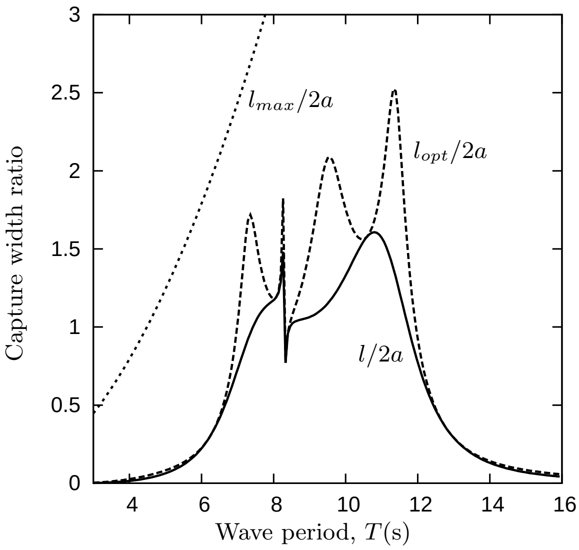

FIGURE 3: Maximum, optimum and achievable capture width ratio for a device of diameter $2a$ =15m, under head seas.

References

1. BRETSCHNEIDER, C.L. 1959. Tech. Rep. U.S. Army Corps of Engineers.
2. CORDONNIER, J., GORINTIN, F., DE CAGNY, A., ET AL. 2015, *Ren. Energy*, **80**, 40–52.
3. CROWLEY, S.H., PORTER, R. & EVANS, D.V. 2013, *J. Fluid. Mech.*, **716**, 566–596.
4. HULME, A. 1982, *J. Fluid. Mech.* **121**, 443–463.
5. NEILSEN, K. & PONTES, T. 2010, *Report T02-1.1 OES IA Annex AA Task.*

Acknowledgments

The authors would like to acknowledge funding from EPSRC grant number EP/N508640/1.