Waves generated by an impulsive perturbation on vertical cylinder

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Transient waves generated by an impulsive perturbation on an infinity vertical circular cylinder are considered. On the cylindrical surface, the perturbation is written by an expansion composed of Fourier series in polar direction and Laguerre function in vertical direction. The waves generated by an elementary Fourier-Laguerre perturbation are then expressed by the memory integration of a double integration of the classical transient Green function associated with an impulsive source which is expressed by a wavenumber integral. The double integral on the cylindrical surface and that along the intersection with mean free surface are performed analytically. Furthermore, the double integrals associated with Fourier-Laguerre base functions by applying Galerkin collocation to the integral equation are carried out also in an analytic way. Numerical results are analyzed and compared with those using classical methods based on point collocation on panels.

1 Introduction

This work is an essential step in the development of a new method based on domain decomposition using a cylindrical control surface at some distance from the body. In the external domain from the control surface, the transient Green function satisfying the boundary condition on the free surface is used while in the internal domain limited by the control surface, the body hull and portion of free surface in between, the simple Green function ($1/r$) is used. Unlike classical methods, the free-surface Green function is not explicitly evaluated but its integration on the cylindrical surface associated with a distribution expressed by Fourier-Laguerre expansion is needed. The integration with the analytical functions on the analytical surface has important properties of excellent convergence and numerical stability.

2 Potential generated by an impulsive perturbation

In the same procedure of linear time-domain analysis to study the seakeeping characteristics of a floating body advancing in waves, as summarized in Beck (1994), we consider a control surface C moving at the same speed U as the body. The surface C is designed to surrounding entirely the body which is located at the center of the reference system with xoy plane on the mean free surface, oz axis positive upwards and ox axis pointing forward. If we assume the radius of control surface and the gravity acceleration are both unit, the speed U is then equivalent to radius-scaled Froude number.

Applying the Green theorem in the external domain limited by the control surface C, the free surface F and a fictitious surface at infinity S_∞, we can write the velocity potential at a point $P(x,y,z)$ by :

$$\Phi(P,t) = \int_0^t d\tau \iint_{C+F+S_\infty} \big[\Phi_n(Q,\tau)G(P,t,Q,\tau) - \Phi(Q,\tau)G_n(P,t,Q,\tau)\big]\, dS(Q) \tag{1}$$

$$= \int_0^t d\tau \iint_C \big(\Phi_n G - \Phi G_n\big)\, dS + \int_0^t d\tau \int_W \big[U^2(\Phi_\xi G - \Phi G_\xi) - U(\Phi_\tau G - \Phi G_\tau)\big]\, d\eta \tag{2}$$

according to Liapis (1986). The integral on S_∞ on the right side of (1) disappears thanks to the property of Φ and the integral on F is transformed to the line integral along the intersection W of the control surface at the free surface in (2) by using Stokes' theorem and the boundary condition on the free surface. The Green function $G(P,t,Q,\tau)$ in (2) is defined in Wehausen & Laitone (1960) as :

$$4\pi G(P,t,Q,\tau) = \delta(t-\tau)G^0 + H(t-\tau)G^F \tag{3}$$

with $$G^0 = -1/|PQ| + 1/|PQ'| \quad \text{and} \quad G^F = -2\int_0^\infty e^{\kappa(z+\zeta)} J_0(\kappa R^t)\sqrt{\kappa}\sin[\sqrt{\kappa}(t-\tau)]\, d\kappa \tag{4}$$

representing the velocity potential at space-time $(P,t) = (x,y,z,t)$ generated by the impulsive source at $(Q,\tau) = (\xi,\eta,\zeta,\tau)$. The part G^0 is the instantaneous term associated with the Delta function $\delta(t-\tau)$. The point $Q'(\xi,\eta,-\zeta)$ is the symmetrical point of Q with respect to $z=0$. The part G^F is called the memory term persisting for $t>\tau$ as indicated by the Heaviside function $H(t-\tau)$. Introducing (3) into (2), we may write :

$$4\pi\Phi(P,t) = \Phi^0(P,t) + \int_0^t [\Phi_C^F(P,t,\tau) + \Phi_W^F(P,t,\tau)]\, d\tau \tag{5}$$

with the instantaneous part

$$\Phi^0 = \iint_C \big(\Phi_n G^0 - \Phi G_n^0\big)\, dS \tag{6}$$

the memory part associated with surface C and the memory part associated with the intersection W

$$\Phi_C^F = \iint_C \big(\Phi_n G^F - \Phi G_n^F\big)\, dS\ ; \qquad \Phi_W^F = \int_W \big[U^2(\Phi_\xi G^F - \Phi G_\xi^F) - 2U\Phi_\tau G^F\big]\, d\eta \tag{7}$$

The last term in above line integral is a compact form derived from performing the partial integration in τ of (ϕG_τ^F) and using the conditions $\Phi=0$ at $\tau=0$ and $G^F=0$ at $\tau=t$. By using the identity :

$$1/|PQ| = \int_0^\infty e^{-\kappa|z-\zeta|} J_0(\kappa R)\, d\kappa = \sum_{\ell=-\infty}^{\infty} \int_0^\infty e^{-\kappa|z-\zeta|} J_\ell(\kappa h) J_\ell(\kappa h') e^{i\ell(\varphi-\varphi')}\, d\kappa \tag{8}$$

in which R the horizontal distance between P and Q, and the polar coordinates ($h=\sqrt{x^2+y^2}$, $\varphi = \tan^{-1}(y/x)$) and ($h'=\sqrt{\xi^2+\eta^2}$, $\varphi'=\tan^{-1}(\eta/\xi)$) as shown on Figure 1 are used, we can rewrite (6) for Φ^0 as :

$$\Phi^0 = \sum_{\ell=-\infty}^{\infty} \int_0^\infty d\kappa \iint_C [\Phi_n J_\ell(\kappa h') - \Phi\, \kappa J_\ell'(\kappa h')][e^{\kappa(z+\zeta)} - e^{-\kappa|z-\zeta|}] J_\ell(\kappa h) e^{i\ell(\varphi-\varphi')}\, dS \tag{9}$$

The horizontal distance R^t (Figure 1) in the memory term G^F (4) is defined by :

$$R^t = \sqrt{[(x-\xi+U(t-\tau)]^2+(y-\eta)^2} = \sqrt{(h^t)^2+(h')^2-2h^t h'\cos(\varphi^t-\varphi')} \tag{10}$$

in which

$$h^t = \sqrt{[x+U(t-\tau)]^2+y^2} = \sqrt{h^2[1+2v^t\cos\varphi+(v^t)^2]} \quad \text{with} \quad v^t = U(t-\tau)/h \tag{11}$$

and the polar angle φ^t in (10) are given by $\tan\varphi^t = y/[x+U(t-\tau)] = \sin\varphi/(\cos\varphi+v^t)$.

In the same way as in (8), we use the identity :

$$J_0(\kappa R^t) = \sum_{\ell=-\infty}^{\infty} J_\ell(\kappa h^t) J_\ell(\kappa h') e^{i\ell(\varphi^t-\varphi')} \tag{12}$$

in the memory term G^F in (4) and we can write the memory parts Φ_C^F and Φ_W^F as :

$$\Phi_C^F = -2\sum_{\ell=-\infty}^{\infty} \int_0^\infty \sqrt{\kappa}\sin[\sqrt{\kappa}(t-\tau)]d\kappa \iint_C [\Phi_n J_\ell(\kappa h') - \Phi\kappa J_\ell'(\kappa h')] e^{\kappa(z+\zeta)} J_\ell(\kappa h^t) e^{i\ell(\varphi^t-\varphi')}\, dS \tag{13}$$

$$\begin{aligned}\Phi_W^F = -2\sum_{\ell=-\infty}^{\infty} \int_0^\infty \sqrt{\kappa}\sin[\sqrt{\kappa}(t-\tau)]d\kappa \int_W \Big\{ & U^2\Phi_\xi J_\ell(\kappa h') - U^2\Phi\big[\kappa\cos\varphi' J_\ell'(\kappa h') + i\ell\sin\varphi' J_\ell(\kappa h')/h'\big] \\ & - 2U\Phi_\tau J_\ell(\kappa h')\Big\} e^{\kappa z} J_\ell(\kappa h^t) e^{i\ell(\varphi^t-\varphi')} d\eta \end{aligned} \tag{14}$$

On the control surface $C(h'=1, -\pi \le \varphi' < \pi)$, we assume that $\Phi(Q,\tau)$ and $\Phi_n(Q,\tau)$ can be expanded by Fourier-Laguerre series like :

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} \phi_{mn}(\tau)\mathcal{L}_m(-\zeta)e^{in\varphi'} \quad \text{and} \quad \Phi_n = \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} \psi_{mn}(\tau)\mathcal{L}_m(-\zeta)e^{in\varphi'} \tag{15}$$

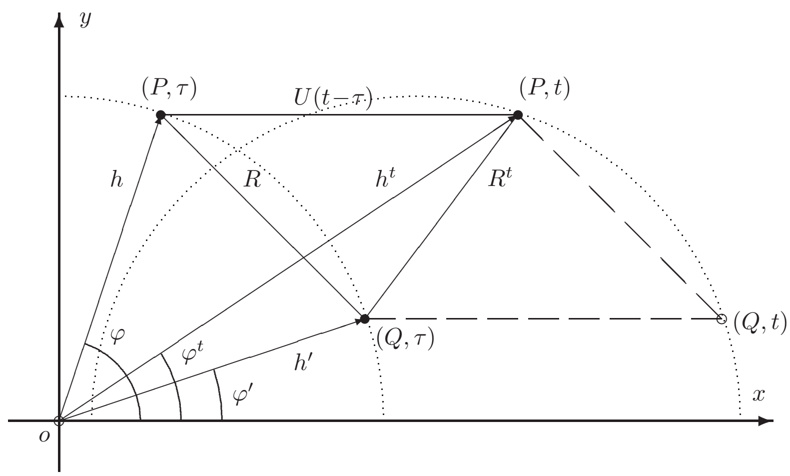

Figure 1: Definition of different geometrical notations

with the Laguerre function $\mathcal{L}(v)$ defined by

$$\mathcal{L}_m(v) = e^{-v/2} L_m(v) \tag{16}$$

with $L_m(v)$ for $v \geq 0$ standing for the mth-order Laguerre polynomial defined in Abramowitz & Stegun (1967).

Introducing (15) into (9), (13) and (14), we may write the potential (5) as :

$$\Phi = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \psi_{mn}(t)\mathcal{G}^0_{mn} - \phi_{mn}(t)\mathcal{H}^0_{mn} + \int_0^t \Big\{ \psi_{mn}(\tau)\big[\mathcal{G}^C_{mn} + \mathcal{G}^W_{mn}\big] - \phi_{mn}(\tau)\big[\mathcal{H}^C_{mn} + \mathcal{H}^W_{mn}\big] - \phi'_{mn}(\tau){\mathcal{H}'}^W_{mn} \Big\}\, d\tau \tag{17}$$

in which $\phi'_{mn}(\tau) = d\phi_{mn}(\tau)/d\tau$ and the terms defined as wavenumver integral are given by :

$$\big\{\mathcal{G}^0_{mn}, \mathcal{H}^0_{mn}\big\} = -(1/2)\int_0^\infty J_n(\kappa h)e^{in\varphi}\big\{J_n(\kappa), \kappa J'_n(\kappa)\big\}\big[Z_m(\kappa, z) - e^{\kappa z} Z'_m(\kappa)\big]\, d\kappa \tag{18}$$

$$\big\{\mathcal{G}^C_{mn}, \mathcal{H}^C_{mn}\big\} = -\int_0^\infty e^{\kappa z} J_n(\kappa h^t)e^{in\varphi^t}\big\{J_n(\kappa), \kappa J'_n(\kappa)\big\} Z'_m(\kappa)\sqrt{\kappa}\sin[\sqrt{\kappa}(t-\tau)]\, d\kappa \tag{19}$$

$$\big\{\mathcal{G}^W_{mn}, \mathcal{H}^W_{mn}, {\mathcal{H}'}^W_{mn}\big\} = -(1/4)\int_0^\infty e^{\kappa z}\big\{\mathcal{J}^{\mathcal{G}}_n(\kappa, h^t, \varphi^t), \mathcal{J}^{\mathcal{H}}_n(\kappa, h^t, \varphi^t), {\mathcal{J}'}^{\mathcal{H}}_n(\kappa, h^t, \varphi^t)\big\}\sqrt{\kappa}\sin[\sqrt{\kappa}(t-\tau)]\, d\kappa \tag{20}$$

with

$$Z_m(\kappa, z) = \frac{2\kappa \mathcal{L}_m(-z)}{\kappa^2 - 1/4} + \sum_{\alpha=0}^{m-1}\left[\frac{(\kappa+1/2)^\alpha}{(\kappa-1/2)^{\alpha+2}} - \frac{(\kappa-1/2)^\alpha}{(\kappa+1/2)^{\alpha+2}}\right]\mathcal{L}_{m-\alpha-1}(-z) - \frac{(\kappa+1/2)^m e^{\kappa z}}{(\kappa-1/2)^{m+1}} \tag{21}$$

$$Z'_m(\kappa) = \frac{(\kappa-1/2)^m}{(\kappa+1/2)^{m+1}} \tag{22}$$

$$\mathcal{J}^{\mathcal{G}}_n = U^2\big\{2J_n(\kappa)J_n(\kappa h^t)e^{in\varphi^t} + J_{n+2}(\kappa)J_{n+2}(\kappa h^t)e^{i(n+2)\varphi^t} + J_{n-2}(\kappa)J_{n-2}(\kappa h^t)e^{i(n-2)\varphi^t}\big\} \tag{23}$$

$$\begin{aligned}\mathcal{J}^{\mathcal{H}}_n = U^2\big\{&\kappa\big[2J'_n(\kappa)J_n(\kappa h^t)e^{in\varphi^t} + J'_{n+2}(\kappa)J_{n+2}(\kappa h^t)e^{i(n+2)\varphi^t} + J'_{n-2}(\kappa)J_{n-2}(\kappa h^t)e^{i(n-2)\varphi^t}\big] \\ &+ (2n+2)J_{n+2}(\kappa)J_{n+2}(\kappa h^t)e^{i(n+2)\varphi^t} - (2n-2)J_{n-2}(\kappa)J_{n-2}(\kappa h^t)e^{i(n-2)\varphi^t}\big\}\end{aligned} \tag{24}$$

$$ {\mathcal{J}'}^{\mathcal{H}}_n = 4U\big\{J_{n+1}(\kappa)J_{n+1}(\kappa h^t)e^{i(n+1)\varphi^t} + J_{n-1}(\kappa)J_{n-1}(\kappa h^t)e^{i(n-1)\varphi^t}\big\} \tag{25}$$

3 Dirichlet-to-Neumann operator in the external domain

On the control surface $C(h' = 1, -\pi < \varphi' \leq \pi)$, both the velocity potential Φ and the normal derivative Φ_n are unknown and expressed by Fourier-Laguerre series (15). The integral representation of velocity potential $\Phi(P,t)$ is also true for $P \in C(h = 1, -\pi < \varphi \leq \pi)$. Integrating both sides of (15) on C after having multiplied the base function $\mathcal{L}_k(-z)e^{-i\ell\varphi}$, we obtain an integral equation :

$$
\begin{aligned}
2\pi\phi_{k\ell}(t) = & \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} \psi_{mn}(t)\mathcal{G}^0_{k\ell,mn} - \phi_{mn}(t)\mathcal{H}^0_{k\ell,mn} \\
& + \int_0^t \Big\{\psi_{mn}(\tau)[\mathcal{G}^C_{k\ell,mn} + \mathcal{G}^W_{k\ell,mn}] - \phi_{mn}(\tau)[\mathcal{H}^C_{k\ell,mn} + \mathcal{H}^W_{k\ell,mn}] - \phi'_{mn}(\tau){\mathcal{H}'}^W_{k\ell,mn}\Big\}\, d\tau \qquad (26)
\end{aligned}
$$

with the notations :

$$
\begin{aligned}
& \Big\{\mathcal{G}^0_{k\ell,mn}, \mathcal{H}^0_{k\ell,mn}, \mathcal{G}^C_{k\ell,mn}, \mathcal{H}^C_{k\ell,mn}, \mathcal{G}^W_{k\ell,mn}, \mathcal{H}^W_{k\ell,mn}, {\mathcal{H}'}^W_{k\ell,mn}\Big\} \\
& \qquad = \int_{-\infty}^0 \mathcal{L}_k(-z)\, dz \int_{-\pi}^{\pi} e^{-i\ell\varphi} d\varphi \Big\{\mathcal{G}^0_{mn}, \mathcal{H}^0_{mn}, \mathcal{G}^C_{mn}, \mathcal{H}^C_{mn}, \mathcal{G}^W_{mn}, \mathcal{H}^W_{mn}, {\mathcal{H}'}^W_{mn}\Big\} \qquad (27)
\end{aligned}
$$

All integrals on the right side of (27) are performed analytically in Chen (2015) and reduced to single integrals in wavenumber.

4 Discussions

Some terms of $\mathcal{G}^C_{mn}(z, h, \tau)$ defined by (19) are depicted on the left of Figure 2 for ($z = -0.01$, $\tau = 20$) and h varying from 1 to 20, and on the right of Figure 2 for ($z = -0.01$, $h = 2$) and τ varying from 0 to 40. Values at different order ($m = 0 - 3, n = 0$) are shown by colored curves and that of the point Green function G^F is also depicted by the dotted black curve. It is expected that G^F is highly oscillatory with large amplitude and $\mathcal{G}^C_{mn}$ is smooth and tends to zero at large time. It is thus expected that the present work offers a critical element to develop the new multi-domain method.

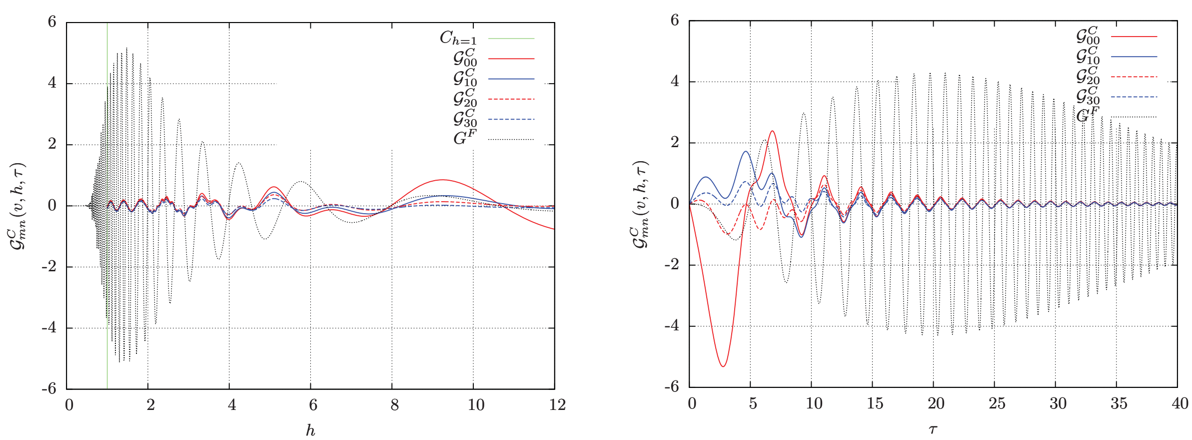

Figure 2: Waves due to an impulsive perturbation at $\tau = 20$ (left) and at $h = 2$ (right)

References

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