Three-dimensional steep wave impact onto a vertical plate of finite width

by

Ioannis K. Chatjigeorgiou[1,2]
i.chatzigeorgiou@uea.ac.uk

Alexander A. Korobkin[1]
a.korobkin@uea.ac.uk

Mark J. Cooker[1]
m.cooker@uea.ac.uk

[1]School of Mathematics, University of East Anglia, Norwich NR4 7TJ, UK
[2]School of Naval Architecture and Marine Engineering, National Technical University of Athens, 9 Heroon Polytechniou Ave, 15773, Athens, Greece

1 Introduction

In this talk we present new investigations of the impulsive force exerted by a steep wave breaking against a vertical structure. We do this by first considering impact against a rectangular vertical plate. Our interest is in the velocity potential (and hence the force) for the fully three-dimensional (3D) short-time fluid dynamics of impact. Under two extreme conditions of the plate's aspect ratio, we show that the 3D solution can be estimated well by two different 2D approximations (strip theory): the comparison helps us to identify those circumstances when strip theory is useful or not.

The breaking wave occupies a constant depth, H, of water, as shown in Fig. 1. On $x = 0$ the plate occupies the rectangular region $-L < y < L$ and $-H < z < 0$, and the remainder of the strip $x = 0$, $|y| > L$, is left as a free surface. The velocity potential φ, satisfies a 3D mixed BVP, in which the normal-derivative of φ on the plate is set equal to V, which is the normal velocity component of the wave face just before impact. The solution depends on just one parameter: $h = H/L$, which contains the half-width L, of the plate. We identify $L = L(t)$ with estimates of the half-width of the wetted region of a wave meeting a vertical structure (one that possesses a large radius of curvature). We start the impact with $L(0) = 0$, and the time-dependence of φ enters parametrically by specifying or modelling $L(t)$. As the impact progresses the wetted region grows and we choose $L(t)$ according to the von Karman and Wagner theories of water entry problems: these give us bounds on the true shape of the complicated real wave-structure wetted region. Knowing $\varphi(x, y, z; t)$ allows us to estimate the short-time impulsive pressure distribution, and hence the total impulsive load on the structure, $I(h)$. We compare $I(h)$ with estimates obtained from 2D strip theory. When $h = H/L$ is either very small or large, we show that strip theory is useful, but when $h = O(1)$ we need the fully 3D solution.

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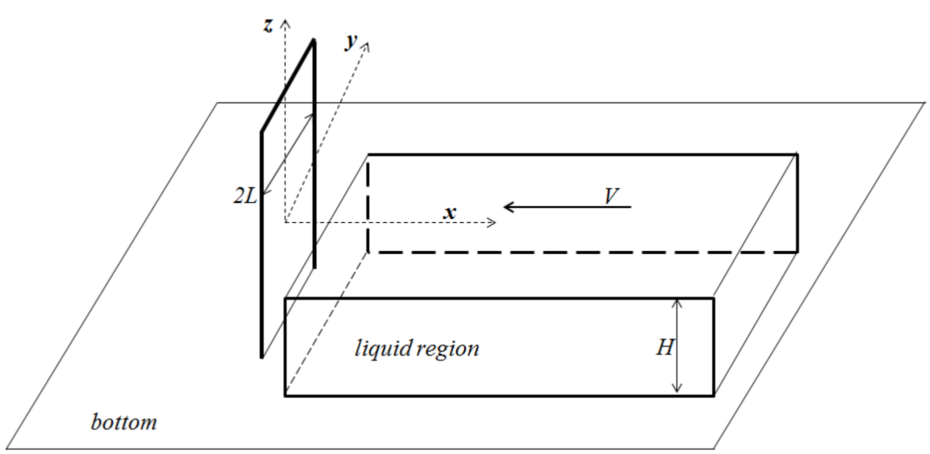

Fig. 1. Configuration of the problem of vertical plate interaction with a steep wave.

2 The mixed boundary value problem

The liquid is inviscid and incompressible and the flow irrotational. We use horizontal coordinates *x*, *y* where $x \geq 0$ and a vertical coordinate *z* such that *z* = -*H* is the bed and *z* = 0 is the free-surface (see Fig. 1). On *x* = 0, the edges of the plate lie at $y = \pm L$. The fluid domain consists of water which approaches the vertical rigid plate at *x* = 0 from the region *x* > 0 and at speed *V* (Fig. 1). We define non-dimensional (~) variables $x = L\tilde{x},\, y = L\tilde{y},\, z = L\tilde{z}$, the change in the velocity potential $\varphi = VL\tilde{\varphi}$ and the fluid pressure $p = -\rho VL\tilde{\varphi}/\Delta t$, where Δt is the short time scale of impact. The tildes are dropped below.

Only the initial stage of the impact is considered, starting from *t* = 0 when the forward front face of the wave first meets the plate. The boundary conditions are linearized and imposed on the position of the undisturbed liquid boundary just before impact. Pressure-impulse theory [1] gives us the dimensionless mixed BVP:

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \quad (x > 0,\quad -\infty < y < \infty,\quad -h < z < 0)\,, \tag{1}$$

$$\varphi = 0 \quad (z = 0,\quad -\infty < y < \infty,\quad x > 0) \text{ and } (x = 0,\quad -h < z < 0,\quad |y| > 1)\,, \tag{2}$$

$$\varphi_z = 0 \quad (z = -h,\quad -\infty < y < \infty,\quad x > 0)\,, \tag{3}$$

$$\varphi_x = 1 \quad (x = 0,\quad -h < z < 0,\quad |y| < 1)\,, \tag{4}$$

$$\varphi \to 0 \quad (x^2 + y^2 \to \infty)\,. \tag{5}$$

Equations (1)-(5) contain only one parameter *h* = *H/L*, the aspect ratio of the plate. Strip theory approaches provide the following 2D approximations for the velocity potential for large *h* >> 1 [2] and very small *h* << 1 [1] aspect ratios on *x* = 0:

$$\varphi^{(h)}(0, y) = -\sqrt{1 - y^2} \quad (|y| < 1)\,, \qquad \varphi^{(v)}(0, z) = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi z/(2h)]}{(2n-1)^2}\,. \tag{6}$$

The superscripts (*h*-horizontal) and (*v*-vertical) correspond to *h* >> 1 and *h* << 1, respectively.

3 Solution of the three-dimensional problem of wave impact and calculation of the total impulse

The solution of equations (1)-(5) is sought in the form

$$\varphi(x, y, z) = -\frac{2}{h}\sum_{n=1}^{\infty}\lambda_n^{-1}\varphi_n(x, y)\sin(\lambda_n z), \quad \lambda_n = \frac{\pi}{2h}(2n-1)\,, \tag{7}$$

where the coefficients $\varphi_n(x, y)$ satisfy the following mixed boundary-value problem of modified Helmholtz type, in which subscripts that are variables, denote partial derivatives:

$$\varphi_{n,xx} + \varphi_{n,yy} - \lambda_n^2\varphi_n = 0 \quad (x > 0)\,, \tag{8}$$

$$\varphi_n = 0 \quad (x = 0,\quad |y| > 1)\,, \tag{9}$$

$$\varphi_{n,x} = 1 \quad (x = 0,\quad |y| < 1)\,, \tag{10}$$

$$\varphi_n \to 0 \quad (x^2 + y^2 \to \infty)\,. \tag{11}$$

The problem of (8)-(11) was considered by Egorov [3]. To find the solution of the 2D problem (8)-(11) we consider the plate -1 < *y* < 1, *x* = 0, as a slender elliptical cylinder. That allows us to consider the problem in elliptical coordinates (*u*, *v*). The boundary condition (9), is thus transformed into $\varphi_n(u,0) = \varphi_n(u,\pi) = 0$ whilst the Helmholtz equation (8) is split into the periodic and radial Mathieu equations. Among the various harmonics that satisfy the Helmholtz equation, only the odd harmonics with odd order satisfy the boundary condition (9). These harmonics are $\varphi_n^{(2m+1)}(u, v) = Ms_{2m+1}^{(3)}(u, -q_n)se_{2m+1}(v, -q_n)$ where $Ms_{2m+1}^{(3)}(u, -q_n)$ are the odd radial Mathieu functions and $se_{2m+1}(v, -q_n)$ are the odd periodic Mathieu functions. Also $q_n = (\lambda_n/2)^2$. The plate condition (10) in elliptical coordinates is $\varphi_u = \sin v, \quad (u=0, \quad 0<v<\pi)$. This is satisfied by considering the series

$$\varphi_n(u,v) = \sum_{m=0}^{\infty} D_{2m+1}^{(n)} \varphi_n^{(2m+1)}(u,v). \tag{12}$$

where $D_{2m+1}^{(n)}$ are coefficients to be found from (10). After some algebra expression (7) for the velocity potential on the plate, is

$$\varphi(0,y,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} C_{mn}(h) se_{2m+1}(v,-q_n)\sin(\lambda_n z), \; C_{mn}(h) = -\frac{4}{(2n-1)\pi} B_1^{(2m+1)}(-q_n)\frac{Ms_{2m+1}^{(3)}(0,-q_n)}{\dot{M}s_{2m+1}^{(3)}(0,-q_n)}. \tag{13}$$

where $B_1^{(2m+1)}$ is the zeroth order expansion coefficient of $se_{2m+1}(v,-q_n)$. We have now obtained the change in the velocity potential during the short time-interval [0, Δt] of impact. The dimensionless pressure-impulse is accordingly just φ. Hence the total impulse on the structure, $I(h)$, is:

$$I(h) = -\int_{-h}^{0}\int_{-1}^{1} \varphi(0,y,z;t)dydz = \frac{4h^2}{\pi^3}\sum_{m=0}^{\infty}\sum_{n=1}^{N} \frac{(-1)^m A_1^{2m+1}(q_n) ce_{2m+1}(0,q_n)}{(n-1/2)^3 \dot{M}s_{2m+1}^{(3)}(0,-q_n)}. \tag{14}$$

Approximations of $I(h)$ are found from (6). These are

$$I^{(h)}(h) \approx \pi h/2 \quad (h\to\infty), \tag{15}$$

$$I^{(v)}(h) \approx 1.0855h^2 \quad (h\to 0). \tag{16}$$

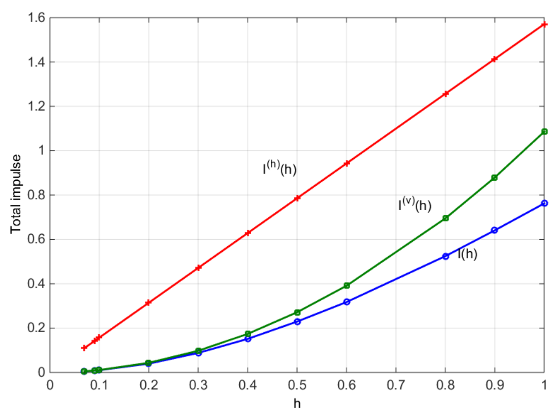

Fig. 2. Total impulse exerted on a vertical plate as a function of the aspect ratio $h = H/L \in [0, 1]$.

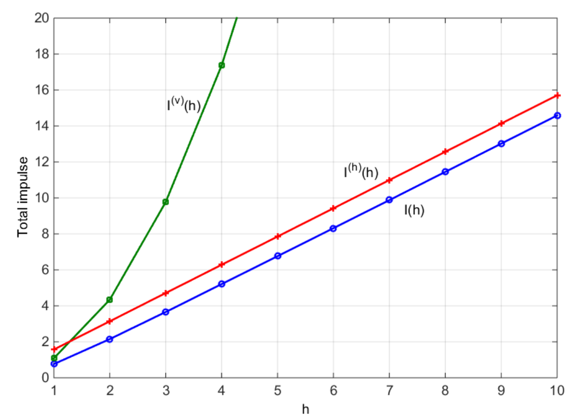

Fig. 3. Total impulse exerted on a vertical plate as a function of the aspect ratio $h = H/L \in [1, 10]$.

The variation of the total impulse as a function of the aspect ratio h, is shown in Figs. 2-3. Strip theory approximations of the total impulse $I^{(h)}(h)$ and $I^{(v)}(h)$ are good only for very large and very small aspect ratios, respectively. As $h \to \infty$, $I^{(h)}(h)$ persists as a constant greater than $I(h)$. This is an end-effect shortcoming in the 2D approximation.

4 Steep wave impact onto a vertical cylinder

The results of the previous section, can be applied to steep wave impact onto a vertical cylinder with radius $R >> L$ and $R >> H$. We do this using both the von Karman and the Wagner approaches for the impacted half-width $L(t)$. Staring from the physical variables, we derive the total force on the cylinder as

$$F_B(t) = -\rho \int_{-H}^{0} \int_{-L(t)}^{L(t)} \varphi_t(0,y,z,t)dydz = -\rho \frac{d}{dt} \int_{-H}^{0} \int_{-L(t)}^{L(t)} \varphi(0,y,z,t)dydz = -\rho \frac{d}{dt}\left[VL^3(t) \int_{-h(t)}^{0} \int_{-1}^{1} \tilde{\varphi}(0,\tilde{y},\tilde{z},t)d\tilde{y}d\tilde{z} \right], \quad (17)$$

In (17) we use the condition that φ is zero on the contact lines $y = \pm L(t)$, $-H \le z \le 0$. The double integral in (17) equals $-I(h)$ [see (14)]. Therefore

$$F_B(t) = \rho V \frac{d}{dt}\left[L^3(t) I(H/L(t)) \right] = \rho V L^2(t) L'(t) \left[3I(h) - hI'(h) \right], \quad (18)$$

where $h(t)= H/L(t)$. The asymptotic behaviour of the hydrodynamic force as $h \to \infty$ $F_B(t) = \pi\rho V L'(t) L(t) H$ whilst the normalized form of (18) is asymptotically as $h \to \infty$, $\hat{F}_B(h) = \left[3I(h) - hI'(h)\right]/(\pi h)$. The hydrodynamic force can be also normalized by $(1/2)\pi\rho V^2 HR$ and yields

$$f_B = \frac{F_B(t)}{(1/2)\pi\rho V^2 HR} = 2\frac{d(L/R)}{d\tau}(L/R), \quad (19)$$

where $\tau = Vt/R$ is the nondimensional time. Fig. 4 shows $F_B(t)$ [normalized by $\pi\rho V L'(t) L(t) H$] as a function of $1/h$. The form of the hydrodynamic force (19) assuming that the contact region is bounded by estimates from the von Karman and the Wagner approaches is shown in Fig. 5. In both cases (19) decreases linearly. The maximum magnitude occurs at the instant of the first impact and for the von Karman case in Fig. 5, f_B=2(1- τ).

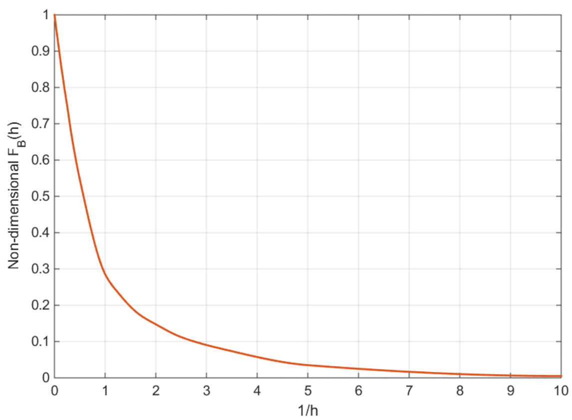

Fig. 4. Non-dimensional hydrodynamic force as a function of $1/h = L/H$.

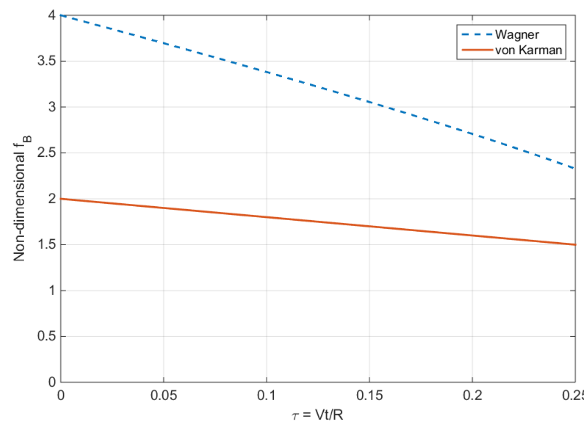

Fig. 5. The non-dimensional hydrodynamic force f_B as a function of $\tau = Vt/R$.

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References

1. Cooker MJ, Peregrine DH. 1995 Pressure-impulse theory for liquid impact problems. J Fluid Mech., 297, 193-214.
2. Wagner H. 1932 Über Stross- und Gleitvorgänge an der Oberfläche von Flüssigkeiten, Z. Angew Math. Mech., 12, 193-215.
3. Egorov IT. 1956 Impact on a compressible fluid. NACA TM 1413 (Translation of "Udar o szhimaemuiu zhidkost"). Prikladnaia Matematika i Mekhanika, 20, 67-72.