Fully nonlinear and dispersive of nearshore wave modeling: accuracy and efficiency of two methods of solving the potential flow problem

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Highlights

- A finite difference approach and a spectral approach of resolving the velocity potential in the vertical are compared in a fully nonlinear potential flow model. The spectral approach shows improved accuracy and efficiency for two test cases.
- The resultant model simulates well the propagation and nonlinear interactions of irregular waves over a submerged bar in comparison to experimental data.

Introduction

To model waves in the nearshore region, it is necessary to have accurate, rapid models that can simulate the nonlinear and dispersive effects over large spatial domains. Models ranging from the mild slope equation to CFD (Computational Fluid Dynamics) approaches based on the Navier-Stokes equations are used for a variety of different applications, with varying degrees of accuracy and computational cost. Potential flow theory models, based on the assumption of irrotational flow, may be an ideal compromise between simplified linear wave models and CFD approaches modeling the fine scale processes.

Potential flow wave models require the resolution of the Laplace equation in the fluid domain with specification of the boundary conditions. Boundary integral methods are commonly used to develop highly accurate fully nonlinear models (e.g. [6]), but recent work also uses a finite difference approach (e.g. [7, 3, 5]). The relative simplicity of this approach, in comparison to more mathematically complex projection methods, may be ideal when studying nonlinear wave-body interactions [3].

Here, a fully nonlinear potential flow theory model resolving the Zakharov equations [12] is developed. The Zakharov equations express the temporal evolution of the free surface elevation η and velocity potential $\tilde{\phi}$, which require calculating the free surface vertical velocity $\tilde{w}$. The accuracy and efficiency of two methods of calculating $\tilde{w}$ as a function of $(\eta, \tilde{\phi})$ ("Dirichlet-to-Neumann" or DtN problem) is compared as a function of the horizontal and vertical resolution for two test cases. The optimal spectral approach is then validated with a comparison to experimental data.

Overview of the mathematical and numerical models

By assuming irrotational flow, the velocity potential $\phi(\underline{x}, z, t)$ satisfies the Laplace equation in the three dimensional $(\underline{x}, z)$ fluid domain:

$$\nabla^2\phi + \phi_{zz} = 0, \qquad -h(\underline{x}) \leq z \leq \eta(\underline{x}, t), \tag{1}$$

with free surface elevation $z = \eta(\underline{x}, t)$ and a bottom boundary $z = -h(\underline{x})$. By assuming continuity of the fluid from the bottom to the free surface (i.e. non-overturning free surface), setting the free surface atmospheric pressure equal to 0, and defining the free surface velocity potential as $\tilde{\phi}(\underline{x}, t) \equiv \phi(\underline{x}, \eta(\underline{x}, t), t)$, the kinematic and dynamic surface nonlinear boundary conditions are derived as a function of $\tilde{\phi}$, following Zakharov [12]:

$$\eta_t = -\nabla\eta \cdot \nabla\tilde{\phi} + \tilde{w}(1 + (\nabla\eta)^2), \tag{2}$$

$$\tilde{\phi}_t = -g\eta - \tfrac{1}{2}\left(\nabla\tilde{\phi}\right)^2 + \tfrac{1}{2}\tilde{w}^2(1 + (\nabla\eta)^2), \tag{3}$$

where $\tilde{w}(\underline{x}, t)$ is the vertical velocity at the free surface defined by:

$$\tilde{w}(\underline{x}, t) = \phi_z(\underline{x}, \eta(\underline{x}, t), t). \tag{4}$$

By specifying the lateral boundary conditions and solving the DtN problem to calculate the free surface velocity $\tilde{w}(\underline{x}, t)$ from $(\eta(\underline{x}, t), \tilde{\phi}(\underline{x}, t))$, (2) and (3) model the temporal evolution of the free surface quantities η and $\tilde{\phi}$.

Equations (2) and (3) are integrated in time using the classical explicit four-step, fourth-order Runge-Kutta scheme. Fourth-order finite difference schemes with regular or irregular point distribution are used to calculate horizontal gradients and Laplacian operators. Two methods of resolving the DtN problem are compared in one horizontal dimension, x, which is discretized by N_X points.

- *Model A.* Following [3], [5], and [4], the domain is discretized with N_Z points in the vertical ($N = N_Z - 1$ fluid layers), and fourth-order finite difference schemes are used to resolve the vertical spatial derivatives.
- *Model B.* The second method, based on the spectral approach of [9], expresses the vertical profile of ϕ as a linear combination of Chebyshev orthogonal polynomials of the first kind (N_T is the maximum order of the polynomial). (See [11] for more details).

For both methods, a system of $N_X(N + 1)$ linear equations must be solved at each time step, where $N = N_L$, the number of layers in the vertical for Model A, and $N = N_T$, the maximum order Chebyshev polynomial for Model B. The direct solver MUMPS ("MUltifrontal Massively Parallel Solver", v4.10.0) [1] is applied in the Fortran code using the default settings.

Test Cases: comparison of accuracy and efficiency

Propagation of a regular nonlinear wave

The first test case compares the resolution of the DtN problem and errors in propagating a regular, nonlinear wave of permanent form in a uniform depth periodic domain. The initial conditions are calculated using a highly accurate Fourier series approximation (20th order) of the stream function method [8] for a wave with wavelength $L = 64$ m and wave height $H = 6.4$ m in a water depth $h = 64$ m (domain length L). The wave steepness is $H/L = 0.1$ (or $ka = kH/2 = \pi/10$), and the relative water depth is $h/L = 1$ (or $kh = 2\pi$).

The normalized error of the free surface vertical velocity decreases with an increase in the vertical resolution N, and both models converge to the same minimum for large values of N (Figure 1). Model

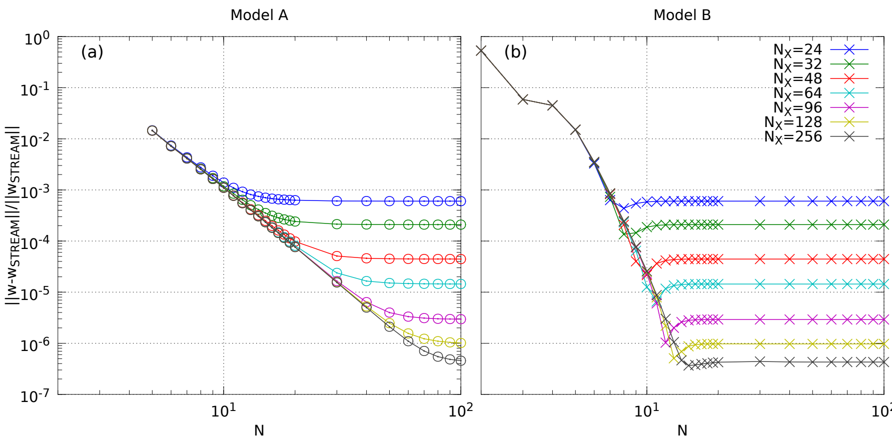

Figure 1: Convergence of the free surface vertical velocity $\tilde{w}$ for a regular nonlinear wave with wave steepness $H/L = 0.1$ ($ka = \pi/10$) and relative water depth $h/L = 1$ ($kh = 2\pi$) for (A) Model A and (b) Model B.

A converges algebraically with errors decreasing as N^{-k}, with $k \approx 3.8$, while Model B converges geometrically, with errors decreasing as $exp(-qN)$, with $q \approx 1.26$. Propagation errors in maximum free surface elevation and phase lag after 25 periods of wave propagation show a strong dependence on the horizontal and vertical resolution. Errors are similar in both model approaches, with expected increases in errors with decreases in resolution, and the optimal value of vertical resolution N for both models appears to be in the range $7 < N < 15$. Comparisons of the computational time required to obtain certain thresholds in total energy errors also shows improved efficiency using the Model B approach.

Motion of a nonlinear standing wave

The second test case compares the motion of a nonlinear standing wave in a domain with fully reflective lateral boundaries. After an integer number of wave periods, the wave characteristics should remain identical to those of the initial condition, calculated using the highly accurate (14th order) Fourier method of [10]. A wave with wavelength $L = 192$ m, relative water depth $kh = 3$, and wave steepness $ka = 0.42$ is calculated (with a corresponding water depth and wave height of $h \approx 91.6732$ m and $H \approx 25.6685$ m, respectively). The simulation is initiated with the displacement of the free surface η in a domain of length L, with no initial fluid velocity.

After 100 periods of wave motion, errors in the maximum free surface position primarily increase with increasing CFL number and decreasing horizontal resolution. Overall, Model B generally produces smaller free surface position errors than Model A, with the exception of the coarsest horizontal grid. Errors in the total energy also increase with increasing CFL number and decreasing horizontal resolution. For small CFL numbers, Model B has overall smaller errors than Model A. For large CFL numbers, the two methods converge to the same errors.

Validation case: propagation of irregular nonlinear waves over a bar

Finally, the selected approach, Model B is validated by simulating the propagation of irregular waves over a submerged bar, reproducing a non-breaking flume experiment of [2]. In the experiments, waves were generated using a piston-type random wave-maker with a JONSWAP spectrum with a peak enhancement factor of $\gamma = 3.3$, with a significant wave height of $H_{m0} = 0.34$ m and a peak period of $T_p = 2.39$ s ($f_p = 0.418$ Hz). The simulation results are compared to the free surface elevation measured at 16 wave probes throughout the domain. In the numerical model, waves are generated in a relaxation zone using the wave spectra calculated at the probe located at the base of the submerged bar, and absorbed in a relaxation zone at the end of the beach.

The simulated wave energy spectra (with $N_T = 7$, $\Delta x = 0.05$ m, and $\Delta t = 0.07$ s) agree well with the experimental data, showing the transfer of energy to super- and sub-harmonics (up to $5f_p$, e.g. Figure 2) as the waves shoal and pass over the trough behind the bar. The spatial variability

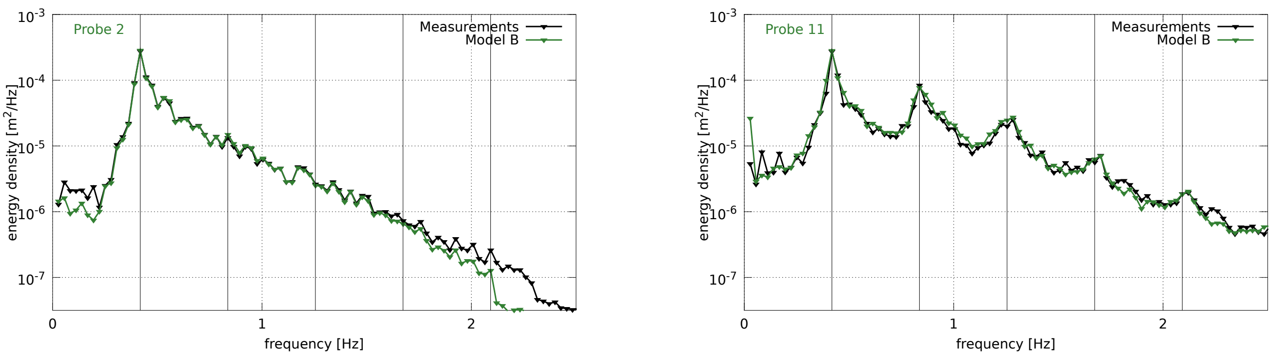

Figure 2: Simulated and measured wave energy density spectra at (left) the base of the bar and (right) the crest of the bar. Vertical black lines indicate the location of the peak frequency and the first four harmonics.

of the integral parameters (significant wave hight, mean period, kurtosis, horizontal and vertical asymmetry) also agrees well with the observations along the bathymetric profile.

Conclusions

Two test cases were used to compare the convergence properties, propagation errors, and CPU time of two approaches to solving the fully nonlinear potential flow problem in 1DH. The Model B spectral approach shows improved accuracy and efficiency in comparison with the Model A fourth-order finite difference schemes. Model B has an exponential convergence rate, while Model A has an algebraic convergence rate, as expected. Based on these tests, the Model B approach was selected and applied to a final validation test case where comparison with experimental data showed its ability to simulate the propagation of irregular nonlinear waves in a wave tank. The optimal value of the vertical resolution, which reduces model errors while limiting the computational time, is recommended in the range of $5 < N_T < 10$ for practical applications with this model. Ongoing work includes the optimization and extension of the Model B approach to 2DH domains.

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