Wave Radiation by a Cylinder Submerged in Water with an Ice Floe or a Polynya

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Highlights:

• Radiation problem for a cylinder submerged in the fluid with a finite elastic plate is solved in two ways: Wiener-Hopf technique and matched eigenfunction expansions.

• The influence of a finite patch of ice-free water in an ice sheet on the hydrodynamic characteristics of oscillating cylinder is investigated.

1. Introduction

The linear 2-D time-harmonic water-wave problem describing small oscillations of a horizontal cylinder is considered for two classes of a hydroelastic system. The fluid surface is either open, except in a finite region where it is covered by a thin-elastic plate, which represents an ice floe, or covered by two semi-infinite thin elastic plates with different properties, except in a finite patch of ice-free water (polynya). In both cases, the fluid domain is of infinite horizontal extent and finite depth.

Radiation of waves by a cylinder submerged in fluid having mixed boundary conditions on the upper surface were studied in [1,2] for a floating semi-infinite elastic plate and in [3,4] for two semi-infinite elastic plates connected by the vertical and flexural rotational springs as a model of a partially frozen crack in ice sheet. These problems were solved by the method of matched eigenfunction expansions for the velocity potentials. The interaction of a submerged body with a floating elastic platform of finite length was considered by Hermans [5] using the Green's function method.

In this paper, Hermans's problem is solved by two different methods: Wiener-Hopf technique and matched eigenfunction expansions. Wave generation by an oscillating submerged cylinder in the presence of a polynya is studied only by using the method of matched eigenfunction expansions. The hydrodynamic load and the amplitudes of vertical displacements of the free surface and elastic plates are calculated.

2. Mathematical formulation

The problem is analyzed in 2-D Cartesian coordinate system with the x-axis directed along the undisturbed mean water surface perpendicular to the cylinder axis, and the y-axis pointing vertically upwards. The fluid is assumed to be inviscid and incompressible, its motion is irrotational. The depth of fluid is equal to H. The plates are in contact with the water at all points for all time. The plate drafts are neglected. It is assumed that the edges of the plates are free.

The wave motions are generated by the small oscillations of submerged rigid body with wetted surface S at a frequency ω with amplitudes ζ_j $(j = 1, 2, 3)$ for the sway, heave and roll problems, respectively. Under the usual assumptions of linear theory, the time-dependent velocity potential can be written as

$$\Phi(x, y, t) = \Re\left[i\omega \sum_{j=1}^{3} \zeta_j \varphi_j(x, y) \exp(i\omega t)\right], \qquad (1)$$

where $\varphi_j(x, y)$ are complex valued functions and t is time. The vertical displacements of the free surface and elastic plates $W(x, t)$ can be determined from the relation

$$\frac{\partial W}{\partial t} = \left.\frac{\partial \Phi}{\partial y}\right|_{y=0}.$$

By analogy with representation (1), the expression for $W(x, t)$ can be written in the form:

$$W(x, t) = \Re\left[\sum_{j=1}^{3} \zeta_j w_j(x) \exp(i\omega t)\right],$$

$$w_j(x) = \left.\frac{\partial \varphi_j}{\partial y}\right|_{y=0}. \qquad (2)$$

The radiation potentials $\varphi_j(x, y)$ satisfy the Laplace equation in the fluid domain

$$\nabla^2 \varphi_j = 0 \quad (-\infty < x < \infty,\ -H < y < 0) \qquad (3)$$

except in the region occupied by the cylinder.

The boundary condition on the closed smooth contour of the submerged body S has the form:

$$\frac{\partial \varphi_j}{\partial n} = n_j \quad (x, y \in S). \qquad (4)$$

Here, $\mathbf{n} = (n_x, n_y)$ is the inward normal to the contour S. The notations

$$n_1 = n_x, \quad n_2 = n_y, \quad n_3 = (y-y_0)n_1-(x-x_0)n_2 \qquad (5)$$

are used where $x_0,\ y_0$ are the coordinates of the center of the roll oscillations.

The boundary condition at the bottom is

$$\frac{\partial \varphi_j}{\partial y} = 0 \quad (-\infty < x < \infty,\ y = -H). \tag{6}$$

In the far field a radiation condition should be imposed that requires the radiated waves to be outgoing.

Finite plate. The upper boundary of the fluid is covered partly with an elastic homogeneous plate $(0 < x < L,\ y = 0)$ with mass density ρ and thickness d. The free surface condition in the open water regions is given by

$$\frac{\partial \varphi_j}{\partial y} - \frac{\omega^2}{g}\varphi_j = 0, \quad (x < 0,\ x > L,\ y = 0), \tag{7}$$

where g is the acceleration due to gravity.

On the elastic covered surface, the radiation potentials $\varphi_j(x, y)$ satisfy the boundary condition in the form

$$\left(D\frac{\partial^4}{\partial x^4} - \omega^2 M + g\rho_0\right)\frac{\partial \varphi_j}{\partial y} - \rho_0\omega^2\varphi_j = 0 \tag{8}$$

$$(0 < x < L,\ y = 0),$$

where $D = Ed^3/[12(1-\nu^2)]$, $M = \rho d$, E is the Young's modulus for the elastic plate, ν is its Poisson's ration, ρ_0 is the fluid density. At the plate edges, free edge conditions require vanishing the bending moment and the shear force:

$$\frac{\partial^3 \varphi_j}{\partial x^2 \partial y} = \frac{\partial^4 \varphi_j}{\partial x^3 \partial y} = 0 \quad (x = 0^+, L^-,\ y = 0). \tag{9}$$

Polynya. Two semi-infinite elastic plates Λ_1 $(x < 0)$ and Λ_2 $(x > L)$ float on water surface. The left plate Λ_1 and the right plate Λ_2 have the characteristics $E_1,\ d_1,\ \rho_1,\ \nu_1$ and $E_2,\ d_2,\ \rho_2,\ \nu_2$, respectively. The boundary conditions for the fluid in contact with the plates Λ_1 and Λ_2 are similar (8) using the corresponding values of $E,\ d,\ \rho,\ \nu$. Free edge conditions (9) are fulfilled at $(x = 0^-, L^+,\ y = 0)$. The free surface condition (7) takes place at $(0 < x < L,\ y = 0)$.

3. Method of solution.

In solving the problem (3), (4), (6)-(9), for each of the body oscillation modes we introduce an unknown mass source distribution $\sigma_j(x, y)$ over the contour S. The radiation potentials at any point of fluid can be represented in the form

$$\varphi_j(x, y) = \int_S \sigma_j(\xi, \eta) G(x, y; \xi, \eta) ds. \tag{10}$$

The Green function $G(x, y; \xi, \eta)$ satisfies the following equation

$$\nabla^2 G = 2\pi\delta(x - \xi)\delta(y - \eta)$$

with the boundary conditions analogous to (6)-(9) and the radiation condition in the far field, and δ is the Dirac delta-function.

We describe briefly the determination of the Green function for the case of finite plate by the Wiener - Hopf technique. The characteristic length $l = g/\omega^2$ and dimensionless variables and parameters are used:

$$x' = \frac{x}{l}, \quad y' = \frac{y}{l}, \quad t' = \omega t, \quad H' = \frac{H}{l}, \quad L' = \frac{L}{l},$$

$$\beta = \frac{D}{\rho_0 g l^4}, \quad \gamma = \frac{M}{\rho_0 l}.$$

Below, the primes are omitted. Then boundary conditions on the upper surface of the fluid (7), (8) have the form

$$\Omega_1(\varphi_j) \equiv \frac{\partial \varphi_j}{\partial y} - \varphi_j = 0,\ (x < 0,\ x > L,\ y = 0), \tag{11}$$

$$\Omega_2(\varphi_j) \equiv \left(\beta\frac{\partial^4}{\partial x^4} + 1 - \gamma\right)\frac{\partial \varphi_j}{\partial y} - \varphi_j = 0, \tag{12}$$

$$(0 < x < L,\ y = 0).$$

Two different expressions for the same Green functions are used. For the first one we seek the Green function in the form

$$G(x, y; \xi, \eta) = G_0(x, y; \xi, \eta) + G_1(x, y; \xi, \eta),$$

where G_0 is the Green function for the fluid with the infinite free upper surface and the condition of non-flow on the bottom, G_1 is the added function to fulfil the conditions (8), (9) on the plate. This expression is convenient when the points (x, y) and (ξ, η) are at short distance, because the logarithmic singularity is expressed evidently. For the determination of the function of G_1 we obtain the same problem (3), (6), (9), (11), (12), only in the condition (12) the right side is non-zero function. This problem is solved by the Wiener - Hopf technique in [6]. For other case we express the Green function in the form of series. We use the Fourie transformation on x

$$\Psi_+(\alpha, y) = \int_L^\infty \mathrm{e}^{i\alpha(x-L)} G(x, y) dx, \tag{13a}$$

$$\Psi_-(\alpha, y) = \int_{-\infty}^0 \mathrm{e}^{i\alpha x} G(x, y) dx, \tag{13b}$$

$$\Psi_1(\alpha, y) = \int_0^L \mathrm{e}^{i\alpha x} G(x, y) dx, \tag{13c}$$

$$\Psi(\alpha, y) = \Psi_-(\alpha, y) + \Psi_1(\alpha, y) + e^{i\alpha L}\Psi_+(\alpha, y),$$

$$\partial^2\Psi/\partial y^2 - \alpha^2\Psi = 2\pi\mathrm{e}^{i\alpha\xi}\delta(y - \eta). \tag{13d}$$

The solution of Eq. (13d) with the condition (6) on the bottom has the form

$$\Psi(\alpha, y) = C(\alpha)Y(\alpha, y) +$$

$$\frac{2\pi}{\alpha}\mathrm{e}^{i\alpha\xi}\begin{cases} \sinh(\alpha(y + H))\cosh(\alpha(\eta + H)) & (y > \eta) \\ \cosh(\alpha(y + H))\sinh(\alpha(\eta + H)) & (y < \eta), \end{cases}$$

$$Y(\alpha, y) = \frac{\cosh(\alpha(y + H))}{\cosh(\alpha H)}.$$

We denote $D_\pm(\alpha),\ D_1(\alpha)$ the integrals of the type (13a-c) where the function G is replaced by the expression $\Omega_1(G)$ and $F_\pm(\alpha),\ F_1(\alpha)$ are analogous integrals where G is replaced by $\Omega_2(G)$. Further the functions $D(\alpha)$ and $F(\alpha)$ are introduced:

$$D(\alpha) = D_-(\alpha) + D_1(\alpha) + e^{i\alpha L} D_+(\alpha),$$

$$F(\alpha) = F_-(\alpha) + F_1(\alpha) + e^{i\alpha L} F_+(\alpha),$$

$$D(\alpha) = \frac{2\pi}{\alpha} \mathrm{e}^{i\alpha\xi}[\alpha\cosh(\alpha H) - \sinh(\alpha H)]\times \cosh(\alpha(\eta + H)) + C(\alpha)K_1(\alpha), \qquad (14)$$

$$F(\alpha) = \frac{2\pi}{\alpha} \mathrm{e}^{i\alpha\xi}[(\beta\alpha^4 + 1 - \gamma)\alpha\cosh(\alpha H) - \sinh(\alpha H)]\times \cosh(\alpha(\eta + H)) + C(\alpha)K_2(\alpha). \qquad (15)$$

From the conditions (11) and (12) we have $D_-(\alpha) = D_+(\alpha) = 0, \quad D_1(\alpha) = D(\alpha), \quad F_1(\alpha) = 0.$

We express $C(\alpha)$ from (14) and substitute to (15). After transformations we obtain

$$F_-(\alpha) + e^{i\alpha L} F_+(\alpha) = \left[2\pi \mathrm{e}^{i\alpha\xi}(\gamma - \beta\alpha^4)Y(\alpha, \eta) + D_1(\alpha)K_2(\alpha)\right]/K_1(\alpha).$$

This equation is solved in a similar manner as in [6]. The solution of this equation and expressions for the Green function are detailed in [7]. An alternative method of determining the Green function is the method of matched eigenfunction expansions which was used in [1-4].

Using boundary condition (4) on the body surface S, we obtain the integral equation for the functions $\sigma_j(x, y)$

$$\pi\sigma_j(x, y) - \int_S \sigma_j(\xi, \eta)\frac{\partial G}{\partial n} ds = n_j.$$

Once the distribution of the singularities $\sigma_j(x, y)$ has been calculated, we can determine the radiation potentials (10). The vertical deflections of the free surface and elastic plates can be determined from (2):

$$w_j(x) = \int_S \sigma_j(\xi, \eta) \frac{\partial G}{\partial y}\Big|_{y=0} ds.$$

The radiation load acting on the oscillating body is determined by the force $\mathbf{F} = (F_1, F_2)$ and the moment F_3 which, without account for the hydrostatic term, have the form

$$F_k = \sum_{j=1}^{3} \zeta_j \tau_{kj} \quad (k = 1, 2, 3),$$

$$\tau_{kj} = \rho\omega^2 \int_S \varphi_j n_k ds = \omega^2\mu_{kj} - i\omega\lambda_{kj}$$

where μ_{kj} and λ_{kj} are the added mass and damping coefficients, respectively. There is the symmetry condition $\tau_{kj} = \tau_{jk}$. Reciprocity relations between the damping coefficients and wave characteristics in the far field agree with the case of infinitely extended free surface (see, *e.g.,* [8]) for a finite elastic plate and with the case of a crack between two semi-infinite elastic plates [3,4] for a polynya.

4. Numerical results

The calculations are performed for the elliptic contour $S\ :\ (x - c)^2/a^2 + (y + h)^2/b^2 = 1$, where a and b are the major and minor axes of the ellipse, respectively, and the coordinates of its center are equal to $x = c,\ y = -h\ (h > 0)$. Rotational oscillations occur with respect to the point $x_0 = 0,\ y_0 = -h$ in (5). The following input data are used: $E = 5GPa,\ \rho = 922.5kg/m^3,\ \nu = 0.3,\ \rho_0 = 1025kg/m^3, d = 2m,\ b = 10m,\ a = h = 20m,\ H = 500m.$

Figures 1, 2 represent dimensionless values of the coefficients of hydrodynamic load as functions of dimensionless frequency $b/l = \omega^2 b/g$:

$$\mu^*_{kj} = \frac{\mu_{kj}}{\pi\rho_0 b^2},\quad \lambda^*_{kj} = \frac{\lambda_{kj}}{\pi\rho_0\omega b^2},$$

$$\mu^*_{k3} = \frac{\mu_{k3}}{\pi\rho_0 b^3},\quad \lambda^*_{k3} = \frac{\lambda_{k3}}{\pi\rho_0\omega b^3} \quad (k, j = 1, 2),$$

$$\mu^*_{33} = \frac{\mu_{33}}{\pi\rho_0 b^4},\quad \lambda^*_{kj} = \frac{\lambda_{kj}}{\pi\rho_0\omega b^4}.$$

More detailed results for the hydrodynamic load on the cylinder and the amplitudes of the displacements of the ice sheets and the free surface will be presented at the Workshop.

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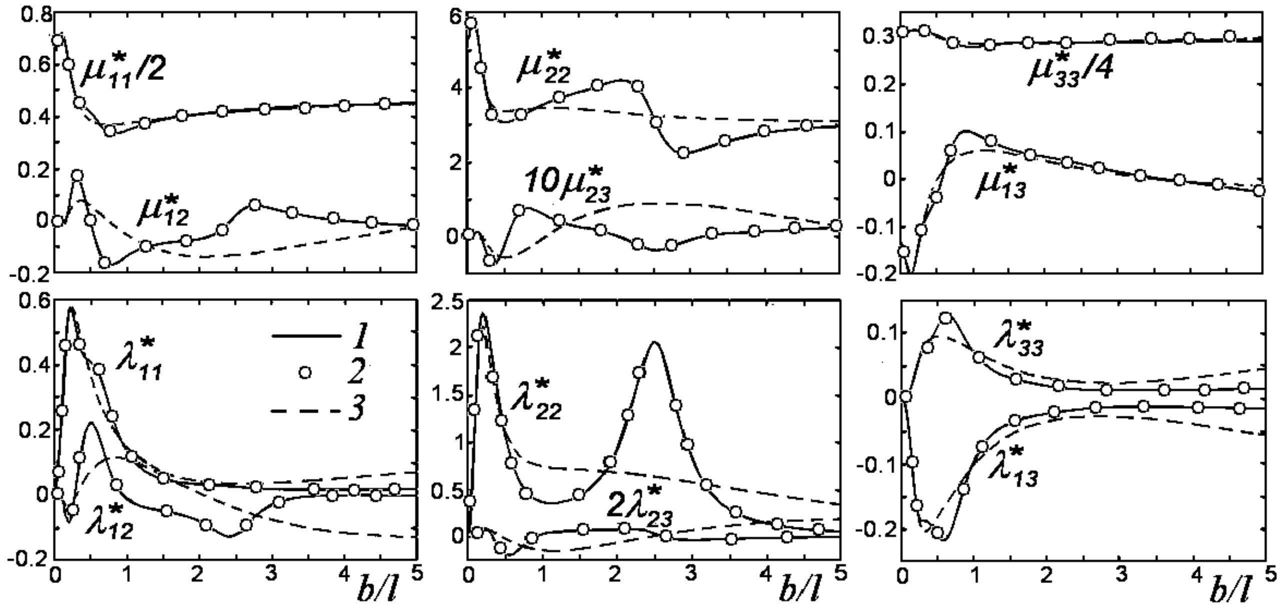

Figure 1: The hydrodynamic load of a cylinder beneath an ice floe at $c/b = 5$. Curves *1* and *2* correspond to solutions obtained at $L/b = 15$ by Wiener-Hopf technique and matched eigenfunction expansions, respectively. Curve *3* shows the results for a semi-infinite ice sheet [2].

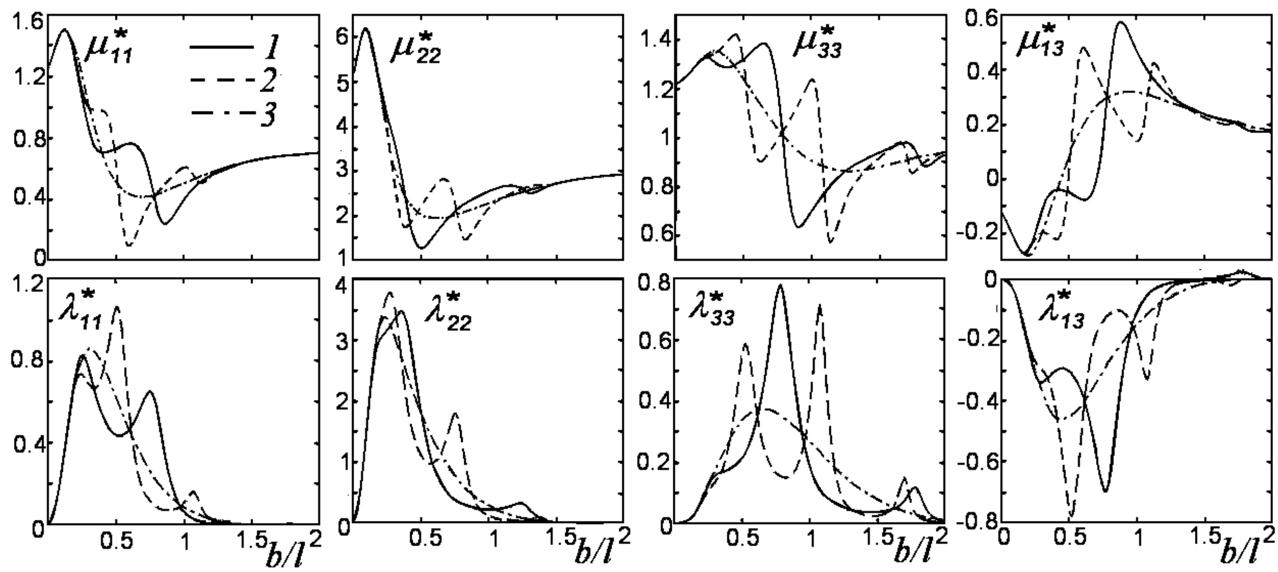

Figure 2: The hydrodynamic load of a cylinder beneath a polynya between two ice sheets with identical properties. Curves *1* and *2* correspond to $L/b = 6,\ c/b = 3$ and $L/b = 10,\ c/b = 5$, respectively. Curve *3* shows the results for infinitely extended free surface.